

# Computer Graphics

## STEM ACTIVITIES WITH TI-NSPIRE

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INSTRUMENTS

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## STEM ACTIVITIES WITH TI-NSPIRE

# COMPUTER GRAPHICS

# INTRODUCTION

This book brings together a number of different mathematical approaches which are used by computer programmers to create and manipulate images. Many of these were originally developed to define shapes used in the manufacture of objects as diverse as drinking glasses, shoes, aircraft wings and car bodies as part of a process called CAD/CAM (Computer Aided Design and Computer Aided Manufacture). More recently many of these techniques have found their way into producing “virtual reality”, initially for simulating views in motion, such as in training pilots to use flight controls to land in a simulator. Now many are used in the entertainment industry to create cartoons, animations, special effects and video games. Most of us are quite unaware just how much mathematics is being used when, for example, you play even the simplest video game – let alone when you “bowl” a ball down a simulated bowling alley on a Nintendo Wii.

The mathematics spans quite a range, from the use of simple constructions, circular arcs, transformation geometry, coordinates and vectors to functions, calculus, matrices, parametric and polar equations etc., and so covers material from KS3, GCSE, AS, A2, FM, S1-S3 and Senior. One way of using the materials is to enhance normal lesson content when a particular topic e.g. vectors is being covered – in order to stimulate some more engagement with the topic by seeing an application. Another is to offer individuals, clubs etc. enrichment activities outside normal lessons e.g. to stretch high attaining students. The materials provide full details of the mathematics involved, as well as references for further reading. However the mathematical details can be skipped by those who just want to use the files to make their own designs.

Most of the geometric (and analytic) techniques used are ones which may be familiar to experienced users of dynamic geometry software, such as Cabri II Plus and the Geometer’s Sketchpad. However we will not assume that most readers of this booklet, whether students or teachers, have that experience, so we also give full details of the software techniques used. Again these can be skipped by those who are already familiar with them.

While the emphasis in this book is on the development and use of TI-Nspire files to create computer graphics, students should also be encouraged to try practical activities such as drawing and making objects, pacing out a locus etc..



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# Chapter 1

Simple animations  
– sliding points  
and setting  
something spinning



Here we start by seeing what sorts of objects we can use as the “home” on which we can set a point and slide it freely about. We’ll start with simple straight objects: segment, vector, ray and line.

First set up a clear Graphs page – to hide the function line at the bottom of the screen, press  $\text{ctrl} + \text{G}$ . **fig\_01**

Click on the Points icon to see the computer drag-down menu, or use  $\text{menu}$  and option 7 on the handheld **fig\_02**. There are four different kinds of straight line objects here: Line, Segment, Ray and Vector. A Segment is defined by 2 points and has a finite length. A Line can also be defined by 2 points and has an infinite length. A Ray can also be defined by 2 points, has an infinite length, but just in one direction – unlike a Line. A Vector can also be defined by 2 points and is finite (like a Segment), but also has a fixed direction (like a Ray).

We will run through how to set up a draggable point on a Segment, and leave it to you to experiment with doing something similar for Lines, Rays and Vectors. As we will see, often there are several ways of achieving the same effect. You may find shortcuts to the ways we suggest – which is fine! In order to define a Segment we are first going to construct, and label, two points called A and B.

First select Point and define two points somewhere on the screen. Use the Text tool from the Actions/Pointer menu to enter the labels A and B (followed by Enter) at the positions of the two points, and press  $\text{esc}$  to Return to Pointer mode. **fig\_03**

Check that you can drag A and/or B about freely, and that the label moves with the point. Now from the Point menu select Segment, Click first on point A and then on point B. Press  $\text{esc}$  to get back to the Pointer mode. Again check that you can drag either end or the whole segment. Now use Point On from the Points menu and click anywhere on the Segment.

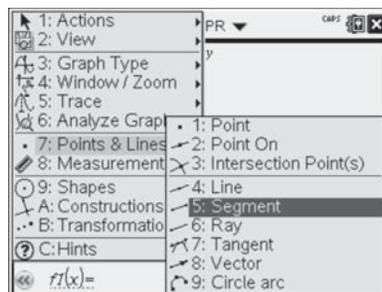
Then right-click ( $\text{ctrl} + \text{menu}$ ) on the handheld) to bring down the menu which allows you to change the properties associated with the new point, such as its Label – then name the point P. Right-click on P again, and select Attributes. Here the drop-down has two options: the shape to use for the point and the animation control. Select the second box and now enter a single digit number, such as 3, then  $\text{enter}$  to set the point P in motion. Press  $\cdot$  and the Animation Controller appears on the screen. Use the left button to Reset the animation so that P starts from its original position, and the right button to set it motion from its current position. **fig\_04**

Now you can try creating a Line, Ray and/or Vector and see if you can construct a point on the object and get it to animate. If the current Graphs view has several points which can be animated, then the Animations Controller applies to them all. You can also create a point on one or other of the axes and also set that in motion.

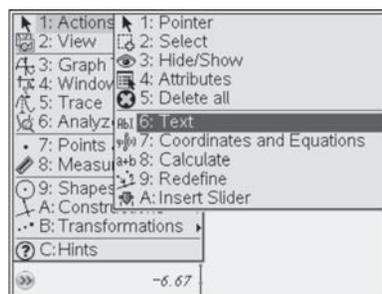
In general we do not use infinite objects, such as Lines, Rays or Axes as the “home” for our animated objects, but stick to finite ones such as Segments and Vectors.



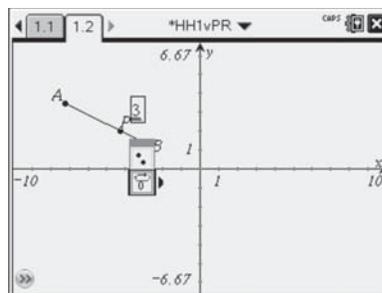
fig\_01



fig\_02



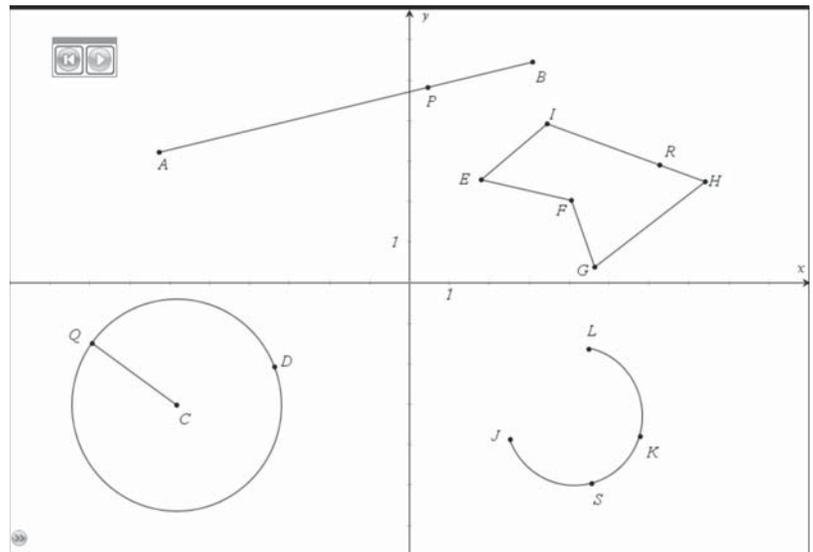
fig\_03



fig\_04

We will now explore slightly more exotic homes for points, such as Polygons, Circles and Arcs.

The screen shows a circle defined by a centre  $C$  and a point  $D$ ; a pentagon defined by points  $E, F, G, H$  and  $I$ ; and a circular arc defined by points  $J, K$  and  $L$ . One point has been defined on each object:  $Q, R$  and  $S$ . Each has been given an animation attribute. So when the animation controller is used to set things in motion, all 4 points move at their various rates. **fig\_05**



**fig\_05**

Later on we shall see how we can also use a variety of curves other than circles as the basis for animations. For now, though, we will see how we can use these animation techniques together with simple geometric constructions and transformations to model a piece of fairground apparatus like a Ferris wheel. The K'nex model in the photograph has 8 seats which hang vertically below 8 points spaced equally around a circle. We will make a simpler model, using just 3 seats - which wouldn't make much money but which should at least illustrate the principles! Of course our model is just a 2D one, unlike the real wheel. **fig\_06**



**fig\_06**

The main features consist of a rigid base (for which we will use an isosceles triangle), the circular rim of the wheel and the three hanging seats. So start a new Problem using a Geometry screen. From the TI-Nspire File menu select Settings and Document Settings ( $\text{doc}$ ) (7) (2) on the handheld). In the dialogue box select Degree instead of Radian for the Angle in the General, Graphs and Geometry pages.

So now we are ready to get going. Start by constructing a Segment  $AB$  across the bottom of the screen. (To make it horizontal hold down the shift key while you draw it.) Then use the Construction menu to select Perpendicular Bisector ( $\text{menu}$ ) (A) (3) on the handheld), and use it to draw the bisector of  $AB$ . Now from the Point menu select Point On Object ( $\text{menu}$ ) (7) (2) on the handheld) and use it to construct the vertex  $C$ . Then select Triangle ( $\text{menu}$ ) (9) (2) on the handheld) and use it to construct the triangle  $ABC$ .

Click anywhere on the triangle and click on the Fill bucket to colour or shade in the triangle (**ctrl** **menu** **B** **enter** on the handheld). So now we have the base for the Ferris wheel and we can adjust its dimensions by sliding A, B or C.

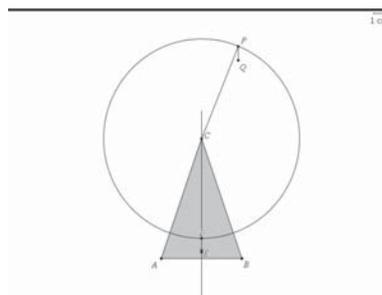
Construct another point D on the bisector below C to define the radius of the wheel and construct the circle (**menu** **9** **1** on the handheld) centre C through D. Construct another point E on the bisector below D to define the length of the seat link DE, and then the vector from D to E (**menu** **7** **8** on the handheld) which will be used to create each new seat. Now construct a point P on the circle, and select Translation from the Construction menu. Click on the point P and the vector DE to translate the point P to the point Q immediately below it, and construct the Segments PQ and CP. Press **esc** and check that as you drag P round the circle so the "seat" Q goes round below it. **fig\_07**

Now we can add in the remaining seats at regular intervals. We could do this by construction using circles to create the other vertices R and T of an equilateral triangle inside the circle. A more general way is to rotate P by 120 degrees around C to get the points R and T. To do this enter a text box (**menu** **1** **6** on the handheld) and enter 120, then use the rotation transformation (**menu** **B** **4** on the handheld) and select the number then the centre of the rotation and finally the point P, then repeat this for your new point. We can then also translate these by vector DE by using the translate tool (**menu** **B** **3** on the handheld) and choosing the point then the vector to create the seats at S and U. Now as you drag P round the points R and S should move in harmony. Can you see what you would need to do to create 5 seats instead? **fig\_08**

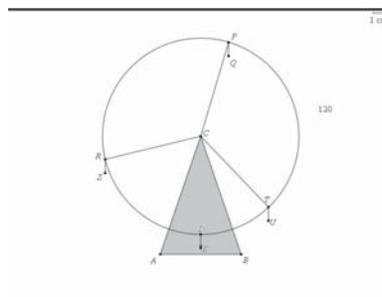
Finally we can set up the Animation Attribute for P, and use Hide/Show to tidy up all but the essential parts of the wheel.

So as well as using geometric constructions we have now used two geometric transformations (rotation and translation) as tools to help with modelling. If we want to add some more interesting "boats" to our Ferris wheel as seats, we can just construct a single one, and use vectors to translate it to each of the desired positions – finally hiding the original drawing and the vectors. **fig\_09**

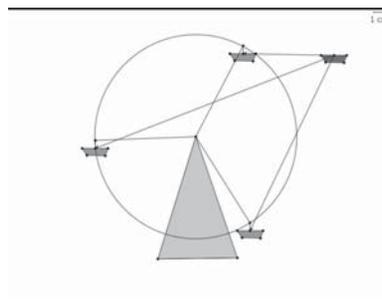
Now see if you can make some animated models of other playground or fairground apparatus, such as swings, roundabouts, slides and see-saws. In each case think first what would be the best object (segment, circle, arc etc.) to use as the "home" for the sliding point.



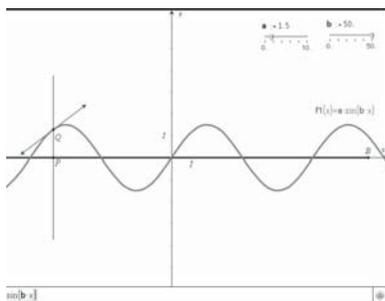
fig\_07



fig\_08



fig\_09

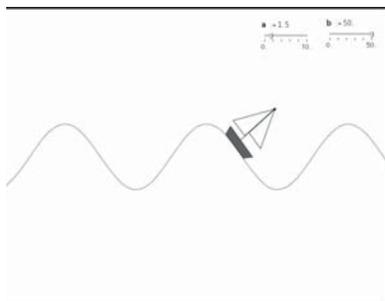


fig\_10

## Can you design a wind turbine with polygonal sails?

So far we have only used geometry, with just a couple of numbers – for animation speed and for rotation angle. If we want to use more complicated shapes than just lines and circles we will need to use some algebra and maybe trigonometry – but in quite a simple way! Suppose we want to model a boat pitching around on a choppy sea – then we could use the graph of a sine wave as the “home” for the sliding point. **fig\_10**

Start a new problem, or a new document, and open a new Graphs page. We can use insert Slider from the Actions menu (ⓀⓂ) on the handheld) to define a variable which we can rename as **a**, and another as **b**. Drag the sliders to somewhere safe. Right-click on the **b** slider and change the Settings so that the maximum is, for example 50. Use the Function option from the Graph menu to enter the formula for the graph:  $y = a \sin(b \cdot x)$ , and play with the sliders to see the effect of changing **a** and/or **b**. Now we could define the sliding point Q to be a point on the graph, but it is an infinite object, so we take a geometric approach instead. Construct points A and B on the x-axis e.g. at (-10,0) and (10,0), and construct the segment AB between them. Now construct P as a Point On AB, and right-click to give it an Animation rate in its Attributes. Press Enter and stop the animation with the Motion Controller. From the Construction menu use Perpendicular to construct a line perpendicular to the x-axis through P. From the Points menu use Intersection Point (ⓂⓃⓅ) on the handheld) to find the point Q where the perpendicular meets the graph. Check that Q slides happily on the crest of the wave as you animate P. (P is an *independent* variable with segment AB as its *domain*, and Q is a *dependent* variable.) From the Points menu select Tangent (ⓂⓃⓇ) on the handheld) to construct the tangent to the graph at Q. Now we have the starting point for building our boat!



fig\_11

When you are happy all is working OK you can use Hide / Show from the Actions menu to hide the axes and labels, as well as the segment AB and the points A, B and P. Use Perpendicular Line from the Construction menu to create the line through P at right angles to the tangent. Use a variety of Construction and Transformation tools, together with shapes like Polygon and Triangle to build your own boat and use Hide/Show to tidy up afterwards once all is working OK. **fig\_11**

If you want to be really imaginative, like modelling the sinking of the Titanic after hitting an iceberg, you will need to find out some more advanced ways of defining graphs – called *piecewise* definitions. The screen below gives a hint of the technique. Ideally we would like some curved arcs to model smooth paths, such as on a roller-coaster. Later sections of this booklet will reveal a number of ways this is done in practice.

## Could you model a Skateboard track?

To conclude this section we look at some other ways to create curved paths and to set things in motion on them. You might like to find out more about mathematical curves such as conics (parabola, ellipse, and hyperbola), cycloids, epicycloids, hypocycloids, limaçons etc.. in books and/or web-sites.

You might already have come across defining curves as loci (the plural of *locus*) – something we can do using geometric constructions. The figure below shows the construction for a parabola – it is the locus of a point Q whose distance QC from a fixed point is equal to its distance QP from a fixed line AB.

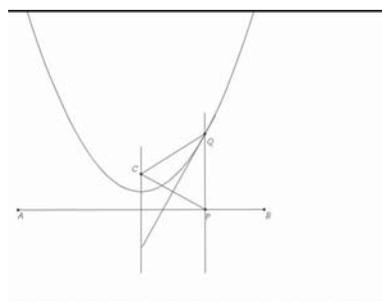
Start with a segment AB, construct its perpendicular bisector and take a point C on it. Construct a sliding point P on AB. Construct the segment PC and the perpendicular to AB through P. Construct the perpendicular bisector of PC and its point of intersection Q with the perpendicular to AB through P. Explain why ABQ must be an isosceles triangle – and hence why  $QC = QP$ . Change the Attributes of P to give an animation speed and so make Q run along the parabolic track. **fig\_12**

As well as graphs of functions like  $y = 5-x$  or  $y = a \cdot \sin(b \cdot x)$ , TI-Nspire lets you draw other kinds of graphs called *parametric* and *polar*. With a parametric graph each point has coordinates  $(x(t), y(t))$  where  $x(t)$  and  $y(t)$  are both functions of another variable (called a *parameter*) – usually called  $t$  because frequently it represents time. Thus for example the path of a basketball during a free-throw has  $x$ - and  $y$ - coordinates which are both functions of the time elapsed,  $t$ . The example on the right shows a Parametric curve defined in Graphs menu (menu 3 2) on the handheld). You enter functions for both  $x$  and  $y$  and also a range of values for the parameter  $t$  – which allows you to define a finite arc, such as of the cubic curve shown – which has an inflection – something we'll see used frequently in computer graphics. If you construct a point on the graph you can animate it and also make it drag its tangent with it.

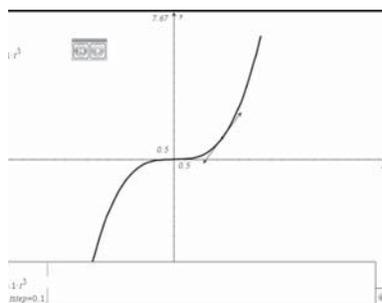
Find out what sort of interesting curves you can make with simple functions for  $x(t)$  and  $y(t)$ . **fig\_13**

The final type of curve we need for now is the *polar* graph (menu 3 3) on the handheld). Imagine a radar scanner rotating around the origin  $O(0,0)$ . The angle  $\theta$  is measured from East anti-clockwise, so N is  $90^\circ$ , W is  $180^\circ$ , S is  $270^\circ$  and E is back to  $0^\circ$ . For any angle  $\theta$  the distance  $r(\theta)$  from the origin is given by some function of  $\theta$ . So for example the polar equation of a circle centre  $O$  radius 3 would be  $r(\theta) = 3$ . We can easily make some more exotic curves using this system, such as the 5-petalled flower on the right. Again, once its graph has been drawn, we can animate a point on it and also find its tangent. **fig\_14**

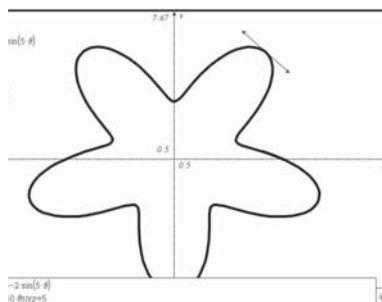
So now you know pretty well all the key techniques for defining curvy paths and putting things in motion on them. All that is needed now are some nice imaginative ideas putting the techniques into practice. Good luck – and do share your results around!



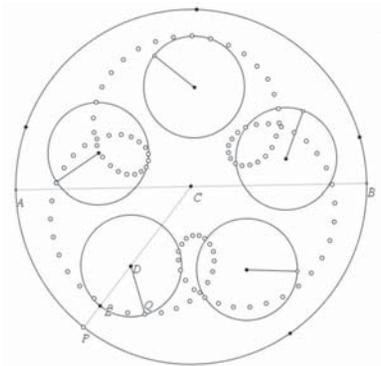
fig\_12



fig\_13



fig\_14



fig\_15

Our final example will be of modelling some combined motion – in this case a fairground ride called the “tea-cup”. The picture on the left shows one rather sedate example. While the base ‘plate’ revolves around the tea-pot, each tea-cup rotates around its own centre at a faster rate. Can you imagine the shape of the path swept out by one of the rider’s feet? **fig\_15**

Our simple model has 5 tea-cups and each cup rotates at rate 3 times faster than the base of the ride. Construct segment AB and its midpoint C to define the circular base. Construct point P on the circle, and the segment PC. Construct D and E as points on PC and construct the circle centre D through E. Construct a point Q on the circle and the segment QD. Change the Attributes of P to give it an animation rate of 1, and of Q to give it a rate of 3. Use Text to enter 72 and Rotate to make 4 copies of the tea-cups at equal spaces.

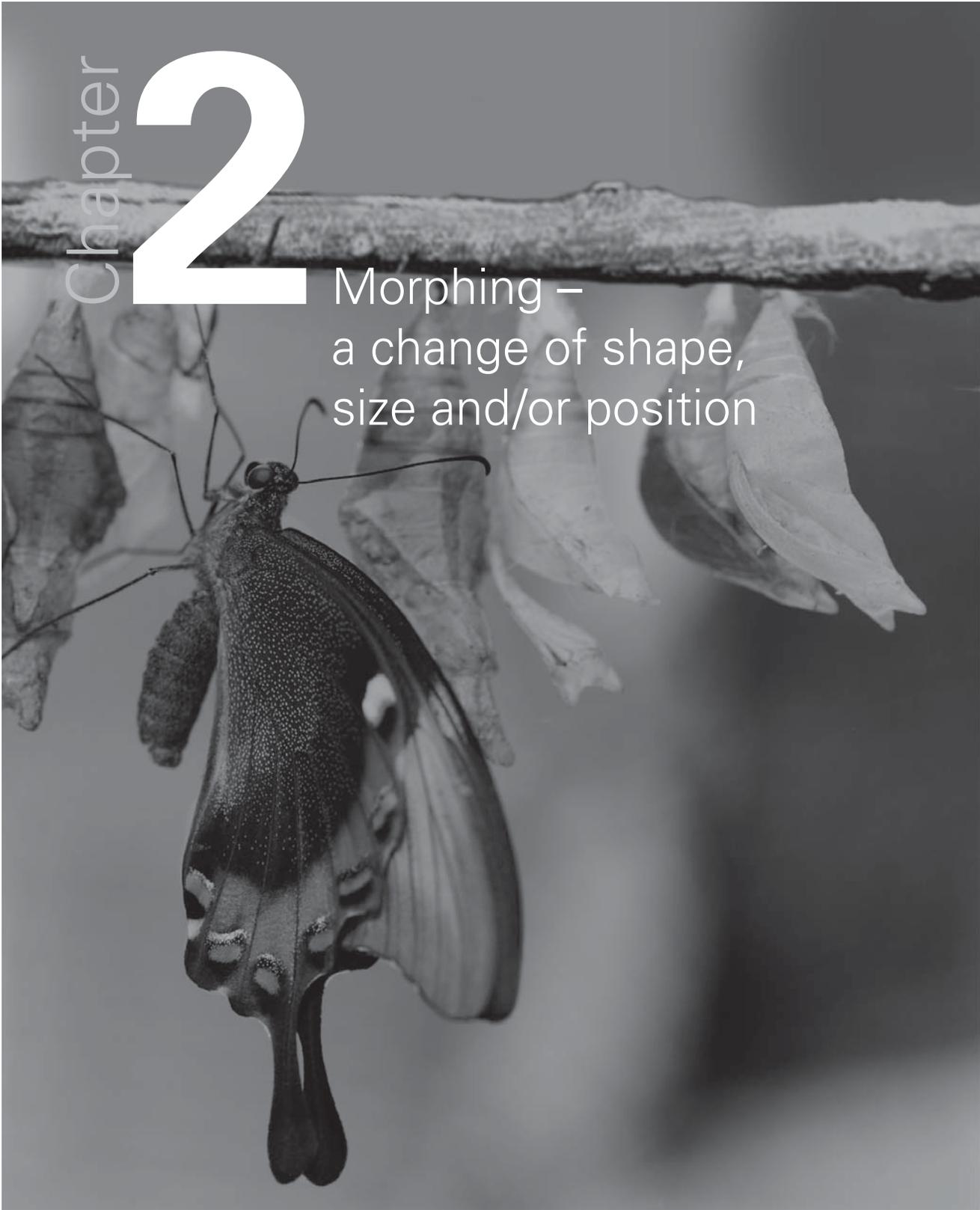
Check that your model animates OK using the Motion Controller. Right-click on Q and select Geometry Trace (⌘ menu 9) on the handheld) to see the path taken by the passenger’s foot.



Chapter

# 2

Morphing –  
a change of shape,  
size and/or position



We are used to seeing very sophisticated video clips where one face or object steadily gets transformed into another, quite different one. While these look pretty tricky they all use some very simple mathematical principles, which are performed many thousands of times in a very short interval to give the impression of smooth changes. The underlying mathematics is based on ratio. Very often we shall have need for a variable which runs just from 0 to 1. The name for such a variable is a **parameter**. The 'meter' part is straightforward – it just means 'a measure'. However 'para' has some quite different meanings – a 'parasol' is 'against the sun', as a 'parachute' is 'against a fall'. But a 'paramedic' is someone 'acting like a doctor', so it's this meaning which probably helps best – a 'parameter' is something 'acting as a measure'.

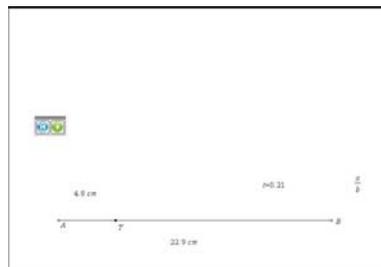
In a new Geometry page hide the Scale ( $\text{MENU}$  2 7). Construct a segment AB and place a point T on it. Measure the distances AT and AB using the measure tool ( $\text{MENU}$  8 1) on the handheld) and then select the length. Use the Text tool ( $\text{MENU}$  1 6) on the handheld) to enter the expression 'a/b' (Use the  $\ominus$  key on the handheld for the '/'), and the Calculate tool ( $\text{MENU}$  1 8  $\text{ENTER}$ ) on the handheld) to substitute AT for 'a' and BT for 'b' in the expression by clicking on the values when prompted.

On the diagram you can see I have used the text tool to add the text 't =' to the resulting value to make it clearer. As you slide T this should vary between 0 and 1. Where does  $t = 0.5$  lie? By right-clicking (/b) on the points the style of A and B have been set to be empty circles, and an animation speed has been set for T – hence the appearance of the animation controller.

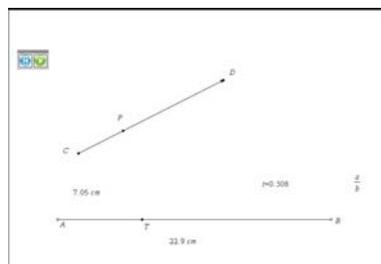
We now have the basic building block for many of the techniques to be applied in this booklet. Remember that T is also an independent variable with the segment AB as its domain – and we use it to compute the value of the variable t which is a parameter with values between 0 and 1.

In our first example we will apply some geometric transformations to translate, rotate and enlarge a shape (polygon) smoothly. Construct a vector CD and dilate (enlarge) about centre C the point D with scale factor the value of t to obtain a point P on it ( $\text{MENU}$  B 5  $\text{ENTER}$ ) then click on C, then the number t and finally D on the handheld). Construct a segment EF and use Compass ( $\text{MENU}$  A 7) on the handheld) to construct a circle centre P with EF as radius. Construct a polygon whose vertices lie on this circle. Check that as you slide T on AB, so P slides on CD, dragging the polygon and circle with it. So we've cracked the translation (aka *slide*) bit!

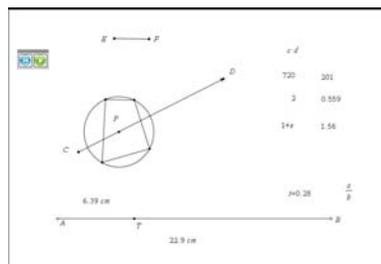
Use the Text tool to enter numbers like 720 and 2 which will be used to control the rotating and enlarging. Check that you have set the angle options in Document Options for Degrees (not Radians). With the Text tool enter the formulae 'c\*d' (Use  $\otimes$  on the hand held for '\*') and '1+e'. Use Calculate to substitute the value 720 for 'c', and that of 't' for 'd'. Check that as you slide T on AB so this result produces numbers from 0 to 720, corresponding to performing two complete rotations in the time T moves from A to B. Now use Calculate again to substitute the value 2 for 'c' and that of 't' for 'd'. Enter the formula '1+e' and Calculate its value with the last result substituted for 'e'. Check that as you slide T on AB that this gives values between 1 and 3, corresponding to increasing enlargement factors up to 3 times in the time taken from T to move from A to B. All we now have to do is to perform



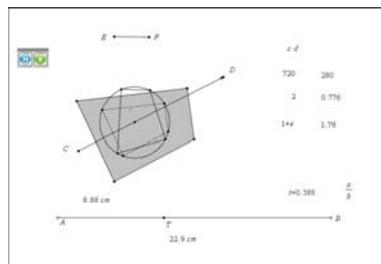
fig\_16



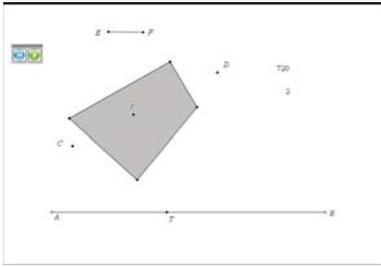
fig\_17



fig\_18



fig\_19

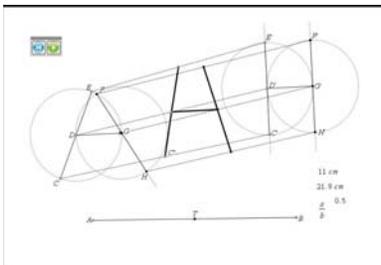


fig\_20

the two transformations and then do some tidying up!

Use the Rotation tool (**menu** **B** **(4)**) to rotate the polygon (shown dotted) around P as centre through the angle currently shown as  $280^\circ$ . Change the Attribute of the resulting polygon to have a dashed perimeter.

Now use the Enlargement tool (**menu** **B** **(5)**) to enlarge the rotated polygon with centre P and scale factor currently shown as 1.78. Change the Attribute of this polygon to shade it in. Now check that as you drag T on AB so the polygon rotates and enlarges!

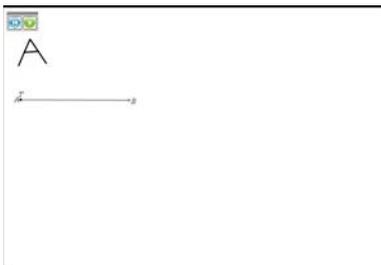


fig\_21

Finally you can hide all the unwanted clutter and use the Start Animation button (which will appear when you click on T) to set it all in motion! It is best to kick it off by first dragging T over A before releasing it.

Making something grow in size is a standard technique to make it appear to be coming towards you – i.e. to use perspective. Of course you may not want your object to follow a straight line path – see if you can use the ideas from Section 1 to replace the vector CD with a curved arc on which P can run.

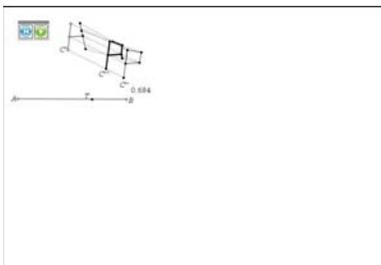
In order to suggest how changing shape is dealt with we shall use a very simple pair of transitions first to map the letter A to H and then to map it to P.



fig\_22

Start a New Problem with a Geometry sheet with the Scale hidden. The idea is that we start with a letter A defined by some points and segments and finish with a letter H. Since H needs 6 points and 5 segments and A needs 5 points and 5 segments we need to use a trick to define A initially using two separate points E and F quite close together and then drag them on top of each other once the transitions have been tested.

Construct the A using points CDEFGH and segments CD, DE, DG, FG and GH. Construct the H using points C'D'E'F'G'H' and segments C'D', D'E', D'G', F'G' and G'H'.



fig\_23

Construct a slider T on a segment AB and use it to calculate the parameter  $t$  between 0 and 1. This will be used to construct '*in-betweenies*' like  $C''$  which is the enlargement, centre C, of the point  $C'$  with scale factor  $t$ . So use the Enlargement tool by selecting the point C, then the Line Segment  $CC'$  and finally the calculated value of  $t$ , repeat this for each of the vertices to construct the 6 *in-betweeny* points like  $C''$  and hence the 5 *in-betweeny* segments forming the shape which will morph from an A to an H. Check it works OK when you slide T. Now you can set up the Animation, drag F on top of E and also hide any unwanted bits.

Now we can construct a (rough!) letter P using 5 points and 5 segments. Next we can set up 6 mappings between points on our current *in-betweeny* H and the letter P. We are going to want the tops of the upward sticks of the H to come together and join up, and also a point on the upper right stick of H to form into the top right of the P – so create an additional point on the H as shown. Now you can construct six segments such as  $C''C'''$  (shown dotted) as the paths taken by the points H as they morph to the P. Construct point  $C''''$  on  $C''C'''$  as the Dilation of  $C''''$  with centre  $C''$  and scale factor  $t$ .

Repeat for the other 5 dotted segments. Join up the points for the new *in-between* (shown with medium weight segments). Check that as you slide T on AB so the new shape closes itself up. Then you can hide all the messy bits and check that the Animation button sets our shape on a path from A to P and back again.

Try some more interesting shapes and paths to sharpen your morphing and creative skills. As a final example we will run through a pleasing kind of kaleidoscope idea based on an equilateral triangle.

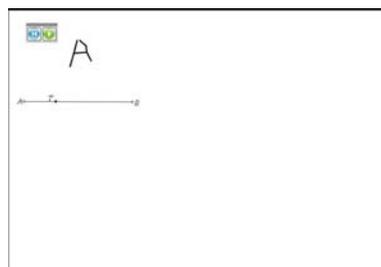
Start a new Problem with a Graphs page. Hide the axes and entry line. We shall often use this as our main modelling page – because it allows us to plot graphs of functions when and where we want and produces the same size page when viewed in TI-Nspire software or on TI-Nspire handhelds. Check that you have Degrees selected both as the Document option and also as the Geometry angle. Construct points O, A, and the circle centre O through A. Construct the intersection points D, E of the circle centre A through O with the first circle.

Construct the intersection point B of the circle centre D through A with the first circle. Construct C similarly. Explain why ABC must be an equilateral triangle (circles are very useful things when it comes to creating equal lengths!) Construct the circular arc through D, O and E and a point P on it.

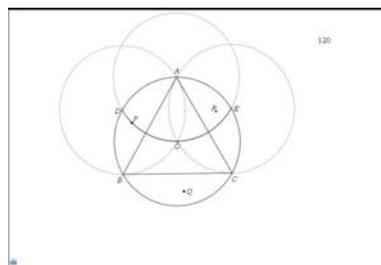
We now want to rotate P around O twice by  $120^\circ$  to obtain points Q and R. Construct the polygon APBQCRA and check that it morphs from a hexagon to a crossed over shape as you slide P. Finally hide all unnecessary clutter and animate P on its arc DOE as domain.

Well now you have met some of the main ideas behind morphing. We need to construct paths between each interesting pairs of corresponding points AA' in the starting and finishing objects. On each path we want to create an in-between point A'' which slides on the arc AA' (which might be a segment, or a circular arc, or part of a curve like those in section 1). We need to control these in-between points so that they each perform their journey from starting to finishing point in the same time. So we will either use one of them to construct all of the others – as in the kaleidoscope idea above – or to have them all controlled by a single parameter t as in the Dilation (aka enlargement) approaches in the earlier examples.

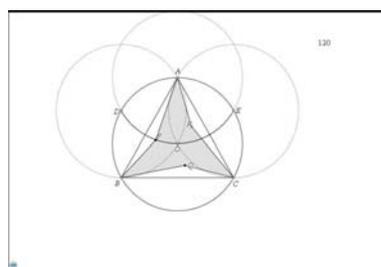
Now it is over to you to see what creative juices flow as you construct your own morphs. Good luck.



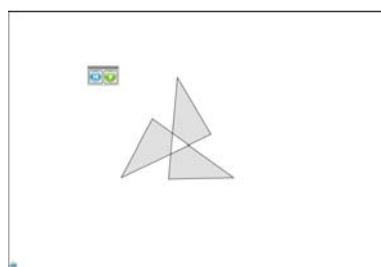
fig\_24



fig\_25



fig\_26

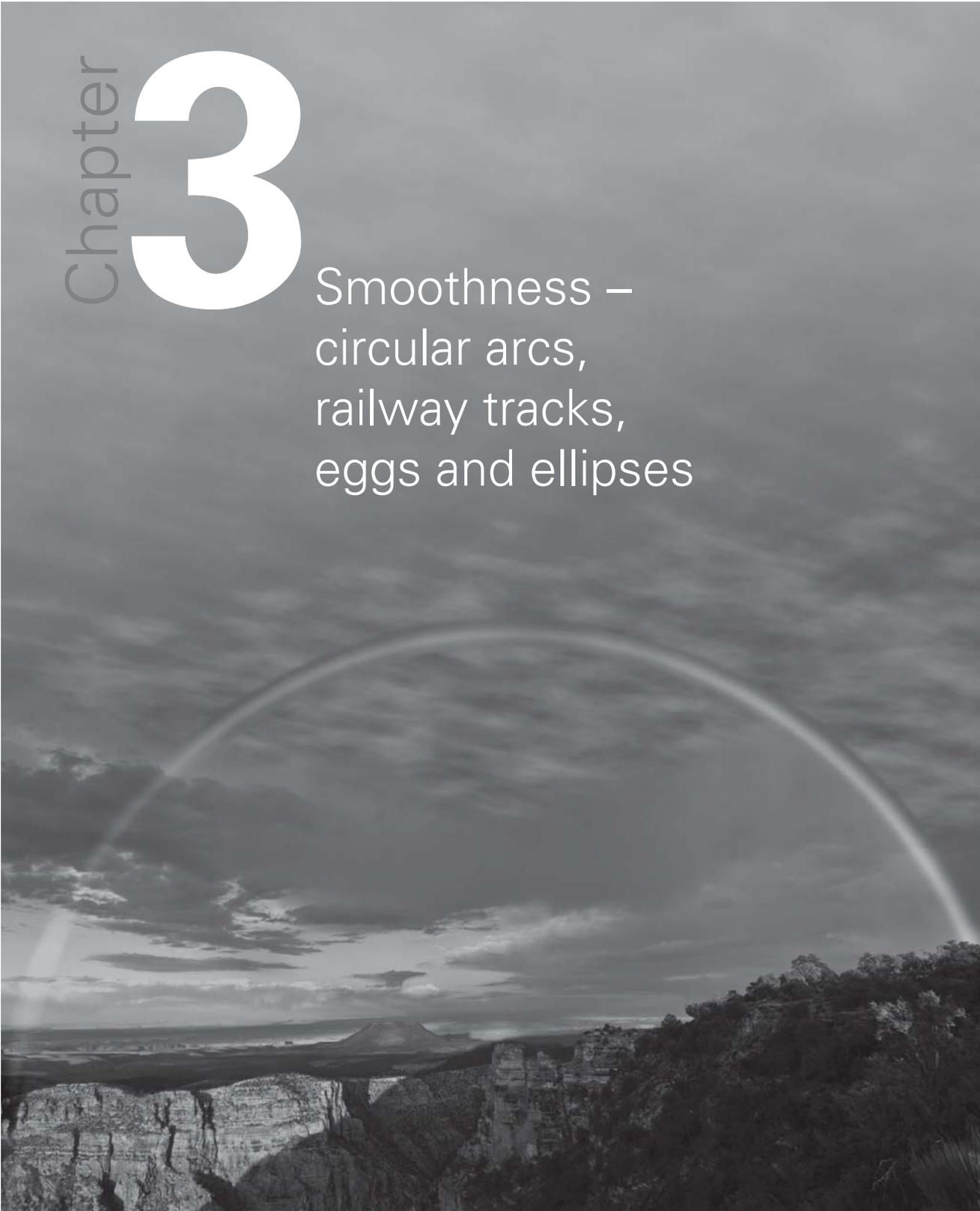


fig\_27

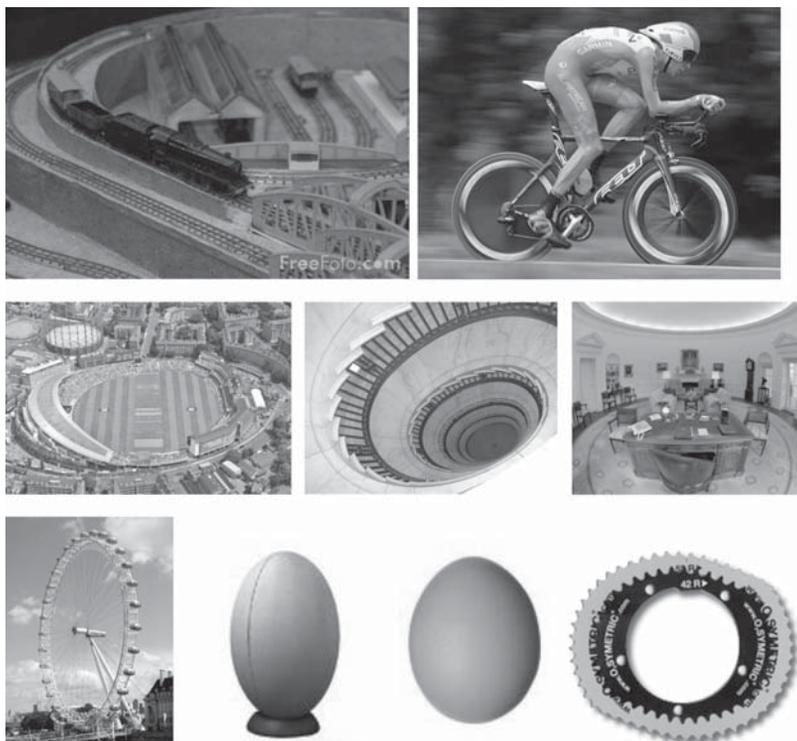
Chapter

# 3

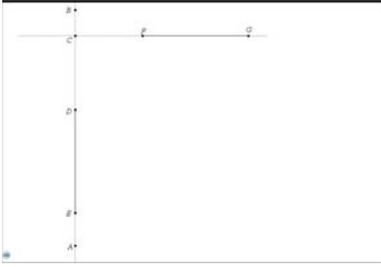
Smoothness –  
circular arcs,  
railway tracks,  
eggs and ellipses



Below are some images which all have something in common – we are looking at things often described loosely as “ovals”. Which would you say are the odd ones out, and why?



The model railway track is made up from pieces which are either straight or arcs of a circle. The British cyclist Bradley Wiggins was photographed in a time trial in the 2009 Tour de France using an odd shaped front chain-ring – also shown above in greater detail (<http://www.osymetric.com/>). The Brit Oval cricket ground in London, the oval staircase in the US Supreme Court building, and the Oval Office in the White House (both in Washington DC, USA) do all seem to share some common property. We know the London Eye must be a circle, but why does the photograph make it look oval? The rugby ball is a 3D object formed by rotating an oval about its longer axis, as is an egg – but are they the same shape? Certainly they all appear round, smooth and non-circular – but there are distinct differences. This section is inspired by the book *Mathographics*, Robert Dixon, Blackwell, 1987 – and it’s all about how to join curves smoothly. It would help if you can imagine trains running along railway lines, turning the handle-bars of a bike to round a bend, or changing direction while skating or skiing. In each case, any change of direction needs to be done smoothly in order to avoid disaster. In section 1 we met a piecewise definition of a function and produced an animation involving some very unsmooth joins!

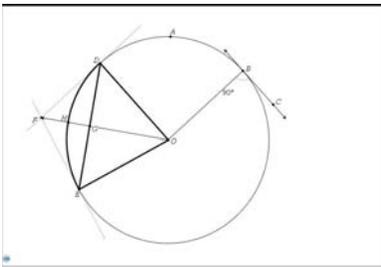


fig\_28

If you wanted to make a  $90^\circ$  change of direction on a railway track it would be no good having two straight sections at right-angles to each other – you would need a curve (probably an arc of a circle) of quite large radius to join them.

Let us see how this might work. Start a new Problem with a Graphs screen, hide the axes and entry line, and set the angle modes to Degrees. Construct Points A and B and the Line AB. Construct a Point C on AB, and use Perpendicular to Construct a line through C perpendicular to AB. Construct Points D and E on AB and the Segment DE. Construct Points F and G on the perpendicular line, and the Segment FG. Can you see how we can draw a circular arc through D and F to make a smooth join between ED and FG? What do you know about tangents to circles? If the arc DF was to be a quarter of a circle how would you find its centre? **fig\_28**

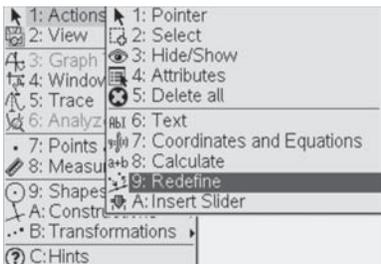
Before we return to this problem we ought to have a new page to use as a check-up on circles and tangents. Construct Points O and A, and the circle centre O through A. Construct a point B on the circle, and the radius OB. Construct the Tangent at B to the circle and a Point C on it. Measure the Angle OBC. Remember this circle fact: the radius and tangent at a point on a circle are perpendicular. Now construct two Points D and E on the circle. The segment DE joining them is a chord of the circle. The tangents at D and E meet at F: what sort of quadrilateral is ODFE?



fig\_29

Can you explain why DF and EF must be equal no matter what the angle DOE? Construct the vector OF (menu 7 8) and its Intersection G with the chord DE. What is the relationship between DE and OF? In order to create a circular arc we need to define three points. Construct the Intersection H of OF with the circle. Now you can use the Circular Arc tool (menu 7 9) with D, H and E to define the Arc DE of the circle. We have used some bold lines to emphasise the 'bow and arrow' shape associated with two points, their chord and their arc.

**fig\_29**

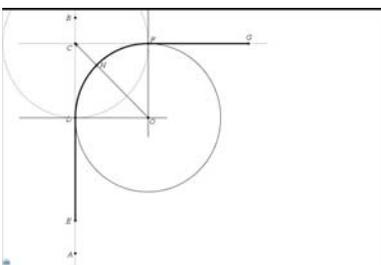


fig\_30

Returning to our original problem you must adjust D, C or F to ensure  $CD=CF$  and then you can construct perpendiculars at D and F to locate the centre of the circle which has DE as tangent at D and FG as tangent at F. Then you can use the bow and arrow trick to construct the Arc joining D to F. Can you make it work?

A useful tool in TI-Nspire is called 'Redefine Point' (menu 1 9) **fig\_30**. So if we construct the Circle centre C through D, then we can Redefine the point F to lie on the intersection of the circle with the perpendicular at C.

The Perpendiculars at D and F meet at O, the centre of the circle which has FG and DE as tangents. Using the bow and arrow construction we can Construct the Vector OC and find its Intersection H with the circle. Then we can Construct the Circle Arc DHF. The smooth railway track is now shown in bold (//b3 then adjust the line weight) as the path EDHFG. Can you imagine cycling along it? What are you doing to the handlebars on ED, at D, on DHF, at F, and on FG? Clearly there will be a maximum speed associated with each such circular curve – the smaller the radius, the slower you can go. **fig\_31**



fig\_31

An alternative measure to the radius at a point is called the **curvature** at the point, where  $\text{curvature} = 1/\text{radius}$ . In this way it makes sense to assign a curvature of 0 to a straight line – and then the bigger the radius of a circular arc, the smaller its curvature. Ideally we want small continuous changes in curvature for a smooth ride – and we will learn more about these in the sections to come.

Before computer graphics became commonplace virtually every company which produced objects had a 'drawing office' where several 'draughtsmen' would produce incredibly accurate drawings and 'blueprints' on drawing boards using geometric instruments such as rulers, compasses, dividers and set squares. So it was very important to be able to create curved objects using just these tools. Also most machine tools for shaping wood, metal, plastics etc. could follow straight or circular paths, but not more complex ones – and so it was important to develop techniques for producing smooth curves using line segments and circular arcs.

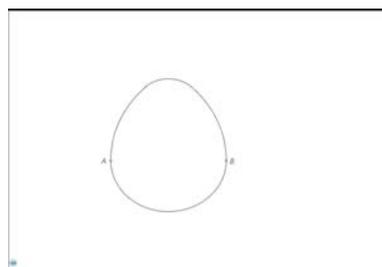
Robert Dixon is a computer graphic artist and a research fellow at the Royal College of Art. One of Dixon's recent claims to fame is in his accusations against the artist Damien Hirst that he copied Dixon's drawings. (<http://www.dailymail.co.uk/news/article-412263/Hirst-accused-plagiarism-artist.html>). In his book *Mathographics*, he discusses a variety of ways of making drawings which model the shapes of birds' eggs. One of the simplest and most pleasing uses just 4 circular arcs and is constructed from a single segment AB.

**fig\_32** is of an 'Egg' constructed by four smoothly joining circular arcs and all based on the single line segment AB. **fig\_33** gives a few clues about how it was constructed. Study it carefully and see if you can work out how to do the construction.

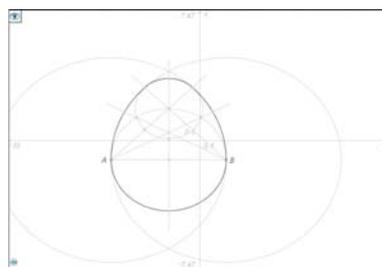
The clue lies in the two large circles. A circle is an important tool because all its radii have the same length.

Here is one way to start the construction. On a clean page construct the segment AB, its midpoint C and the circle centre C through A. Construct the perpendicular bisector of AB and its intersections D and E with the circle. Construct the circle centre B through A. Construct the ray from B through D to meet the large circle in F. Construct the circle centre D through F. Construct the ray AD and its intersection G with the smallest circle. Now you can see one of the arcs BEA. Use the bow and arrow construction to help draw the arc AF. The arc FG is easy to construct – just make an intersection point. **fig\_34**

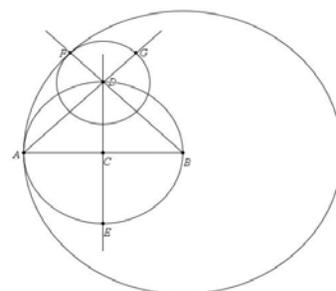
The arc GB just requires a circle to be drawn through G and B, followed by the bow and arrow trick. Then you can hide all the construction work just to show the smooth outline of the 'egg'. While the arcs join smoothly in the sense that they have the same tangent either side of a join, there will be 'jolts' at A, F, G and B if the shape was used as a rail track – at each of these points the **radius of curvature** changes as you move from one arc to the next. This would mean that the flanges of the wheels would bump against the rails to stop the train getting thrown off the track – and if the train was travelling too fast your coffee would shoot out of its container!



fig\_32



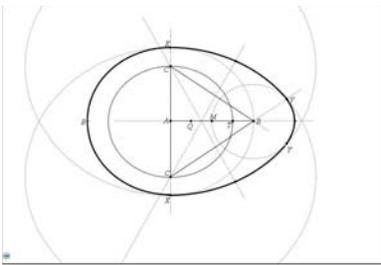
fig\_33



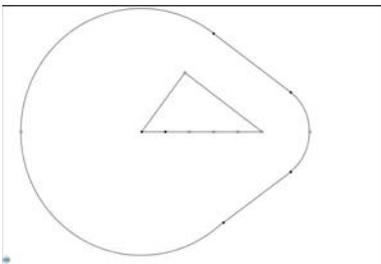
fig\_34



fig\_35



fig\_36



fig\_37

When you've done all the construction, measure the width of the 'egg' AB and its length and hence find the ratio of width to length. Measure some real eggs (don't go birds-nesting!) and see what sorts of ratios occur naturally. Invent some different ways of constructing eggs, or look up some on the Internet or in Dixon's book. Can you calculate the area enclosed by Dixon's egg? How about your own eggs?

This construction also known also as Moss's Egg (<http://mathworld.wolfram.com/MossEgg.html>). It is one of a family also known as Thom's Eggs, after the British engineer Professor Alexander Thom who invented them to model Megalithic stone rings, like the ones at Cairnpapple Hill, Long Meg and Castlerigg. **fig\_35**

The following information is extracted from the Amazon page on the book "Alexander Thom: Cracking the Stone Age Code" (Paperback) by Robin Heath, Bluestone Press, 2007: <http://www.amazon.co.uk/gp/product/0952615142>.

*"Professor Alexander Thom was a foremost scientist and engineer of the last century ... he had been Principal Scientific Officer for the design of the High Speed Wind Tunnel at the Royal Aircraft Establishment, Farnborough, and had assisted Sir Barnes Wallace in the design of the famous 'bouncing bomb' of 'Dambuster' fame. From 1934, Thom became interested in the megalithic culture that had erected the stone circles, rows and other monuments in Neolithic and Bronze Age Britain. He began to accurately survey these sites, and in 1967 published "Megalithic Sites in Britain" (Oxford) where he claimed the builders had been skilled surveyors and astronomers, and had used an identical and accurate unit of length to mark out their constructions throughout Britain, a length he called the Megalithic yard (2.72 feet or 0.829m). Thom also discovered that they were using a geometry based on right-angled 'Pythagorean' triangles, triangles whose sides were whole numbers of this same megalithic yard, or subdivisions or multiples of it."*

You can find out more about his work on Megalithic culture at:

[http://en.wikipedia.org/wiki/Alexander\\_Thom](http://en.wikipedia.org/wiki/Alexander_Thom)

<http://www.dealbhadair.co.uk/athom.htm>

<http://www.megalithic.co.uk/article.php?sid=2146413596>

Thom identified two different egg shapes for stone rings like those at Cairnpapple Hill in West Lothian and others in Devon and Cornwall (<http://mathworld.wolfram.com/ThomsEggs.html>), each based on a 3,4,5 right-angled triangle. See if you can construct his first egg shape from the screen shot **fig\_36**.

See if you can locate a reference to the second egg shape which has 2 circular arcs and 2 straight sections. **fig\_37**

"Long Meg and her daughter" in Cumbria is one of two of the "flattened circle" Megalithic stone shapes identified by Thom. See if you can find and construct either or both of these.

So we have seen some techniques using circular arcs to construct smooth(ish) curves. The next part of this section uses the ellipse as an example of a smooth curve – one which is very important in science, as it describes the paths followed by planets around the Sun and electrons around the nucleus of atoms. While usually considered part of Advanced or Further Mathematics, we can explore most of its properties using simple geometric techniques and some trigonometry.

The following extract from an old book on construction techniques tells a draughtsman how to draw an approximation to an ellipse. See if you can work out how to carry it out in TI-Nspire. We will then see a way to draw an accurate ellipse as a locus and check out how close a match it gives.

*“The four-centre method is used for small ellipses. Given major axis,  $AB$ , and minor axis,  $CD$ , mutually perpendicular at their midpoint,  $O$ , draw  $AD$ , connecting the end points of the two axes. With the dividers set to  $DO$ , measure  $DO$  along  $AO$  and reset the dividers on the remaining distance to  $O$ . With the difference of semi-axes thus set on the dividers, mark off  $DE$  along  $DA$  equal to  $AO$  minus  $DO$ . Draw perpendicular bisector  $AE$ , and extend it to intersect the major axis at  $K$  and the minor axis extended at  $H$ . With the dividers, mark off  $OM$  equal to  $OK$ , and  $OL$  equal to  $OH$ . With  $H$  as a centre and radius  $R1$  equal to  $HD$ , draw the bottom arc. With  $L$  as a centre and the same radius as  $R1$ , draw the top arc. With  $M$  as a centre and the radius  $R2$  equal to  $MB$  draw the end arc. With  $K$  as a centre and the same radius,  $R2$ , draw the end arc. The four circular arcs thus drawn meet, in common points of tangency,  $P$ , at the ends of their radii in their lines of centres.”*

Yes, a picture really is worth a thousand words! With TI-Nspire we use the Compass tool () instead of the draughtsman's dividers, to draw a circle with a given centre and with a radius the same length as a given segment. The figure below shows the full procedure. The four arcs meet in pairs at the points  $P$ ,  $Q$ ,  $R$  and  $S$  – and in each case there is a change of curvature at the join. **fig\_40**

The point  $P$  is the intersection of the upper arc, centre  $L$  and radius  $LP$ , with the right end arc, centre  $K$  and radius  $KP$ , so there is quite a change in radius at  $P$ , and hence curvature. **fig\_41**

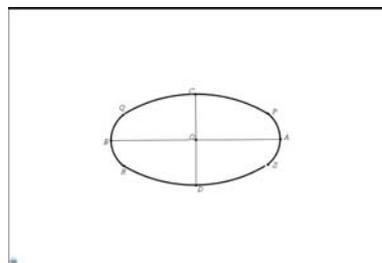
In order to aid readability the screen shots are taken from the higher resolution of the TI-Nspire software display, but all the constructions work fine on the hand-held. In order to explore the actual curvature of a real ellipse we have got to find a suitable way of drawing one which will allow us to construct tangents. Our first attempt uses the construction called ‘auxiliary circles’.



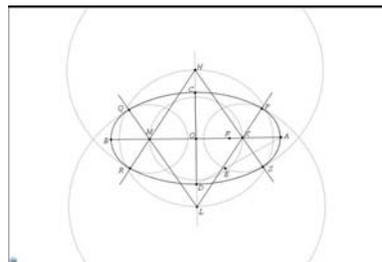
fig\_38



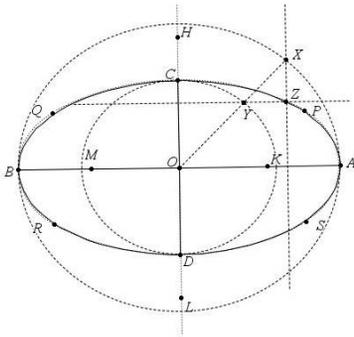
fig\_39



fig\_40

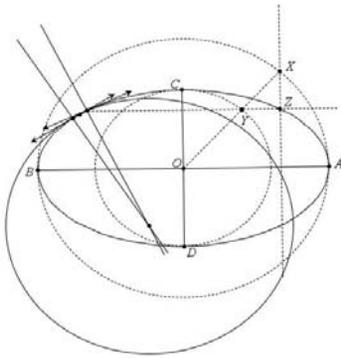


fig\_41



fig\_42

On a new page construct a segment AB, its midpoint O and its perpendicular bisector. Construct the point C on the bisector and reflect it in AB to give the point D. Construct the segment CD. Draw the circle centre O through A (the major circle) and the circle centre O through C (the minor circle). Construct a point X on the major circle and draw the segment OX. Construct the intersection of OX with the minor circle as the point Y. Draw the line through X parallel to CD and through Y parallel to AB. Construct their intersection point Z. Check as you drag X on its circular domain so Z traces out a curve. Use the Locus tool (menu A (6)) to construct the curved path of Z as X slides on the major circle. This locus is an **ellipse**. fig\_42



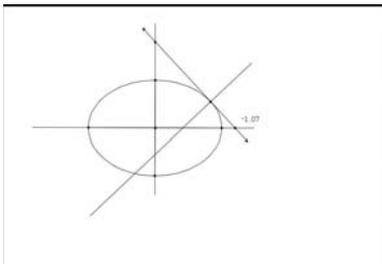
fig\_43

Unfortunately we cannot construct a tangent to the ellipse at the exact point Z. Instead we can construct a point on the locus, draw its tangent and also the perpendicular to the tangent through it (called the **normal** to the ellipse). If we repeat this for another nearby point on the ellipse we will have two normals which intersect in a point. The closer the points are on the curve, the nearer the intersection of the normals is to the centre of curvature to the ellipse. Sadly we can't animate either point so we just have to drag each of them by hand to a new position to see approximately how the curvature changes. In order to animate the points we need to find the equation of an ellipse.

The point Z on the ellipse has the x-coordinate of the point X and the y-coordinate of the point Y. If the angle AOX is taken as the parameter t, and the lengths OA = a, OB = b, then:

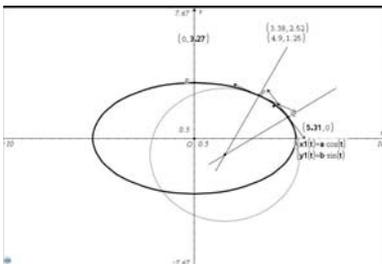
$$x = a \cos(t) \quad \text{and} \quad y = b \sin(t).$$

These are the **parametric equations** of the ellipse as t runs from  $0^\circ$  to  $360^\circ$ . So, on a new Graphs page draw the ellipse as a parametric curve.



fig\_44

To construct this diagram begin by placing a point on the x-axis and find its co-ordinates (menu 1 (7)) and store the x-coordinate as **a** by clicking on the x-coordinate and storing its value. Then repeat this putting a point on the y-axis and storing the y-coordinate as **b**. Now you can plot the parametric equation of the curve (menu 3 (2)) to change to parametric equation mode – and remember to press (x) between a & cos(t) and b & sin(t).

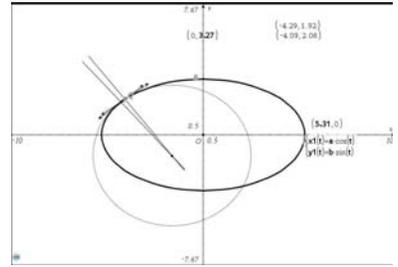


fig\_45

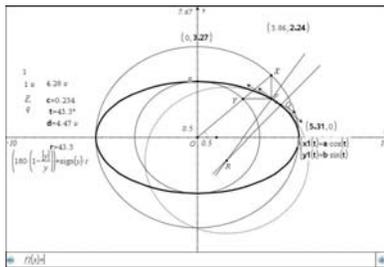
Now we can define points P and Q on the curve, and **YES** we can animate them! So again construct tangents and normals, and hence the approximate centre of curvature. Give them both the same unidirectional animation speed of 1 and pause the animation. Push P and Q very close together. Select Geometry Trace for the point of intersection (menu 5 (4)). Now set the animation going, and watch the approximate centre trace out a shape with four cusps – which is the locus of the centre of curvature – and whose posh name is the **evolute** of the ellipse. See how the shape changes as you slide B to change the **eccentricity** of the ellipse. If  $OB = OA$  the ellipse becomes a circle, the radius of curvature is the same everywhere, and there is a single centre of curvature for all points. The eccentricity is measured by a number  $e = 1 - (b/a)^2$  which is zero for a circle and approaches a maximum of 1 as b tends to zero.

The screens below show an experiment set up to capture data on the curvature at P as the angle  $t$  sweeps from 0 to 360 (it uses some tricks you really do not need to know about just now) – and the resulting graph shows how the curvature moves smoothly between its minimum (straightest) and maximum (sharpest) values. Of course, as  $b$  gets closer to  $a$ , so the ellipse gets closer to a circle and the graph becomes a straight line – corresponding to constant radius.

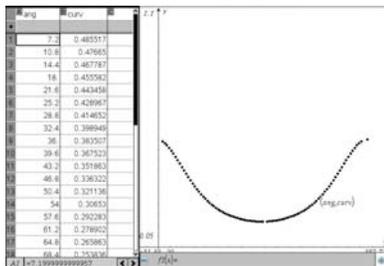
Before we wrap this section up we will just have a look at a couple of properties of the ellipse, which the ancient Greeks studied as a 'conic section'. In the picture alongside, the plane labelled  $\pi$  is a diagonal slice through a cone. It just touches a smaller sphere at  $F$  and a larger sphere at  $F'$ . The cut face is an ellipse having the two points  $F, F'$  as foci. We shall see how they come into play.



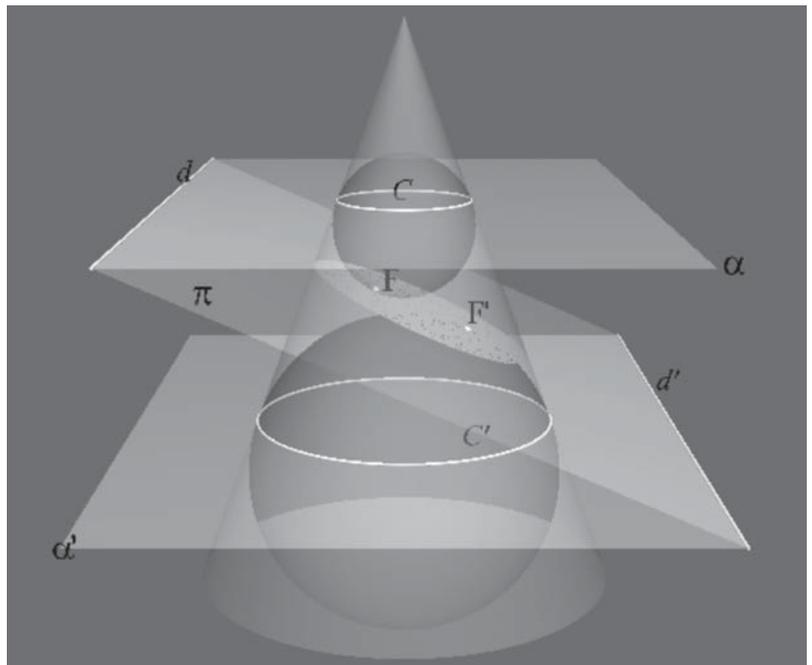
fig\_46



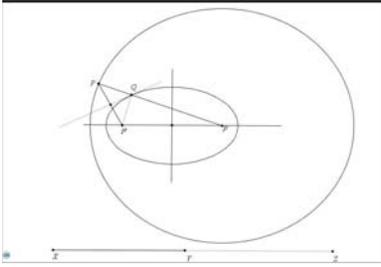
fig\_47



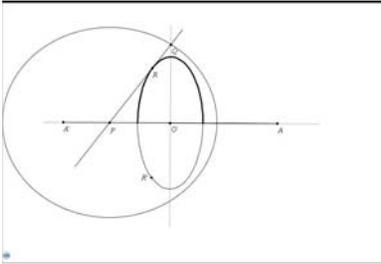
fig\_48



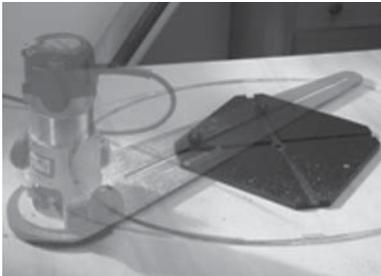
fig\_49



fig\_50



fig\_51



fig\_52



fig\_53

The first property is that the sum of the distances from the foci  $F, F'$  to any point  $Q$  on it is constant – and this is the basis of the “string and two pins” method of construction. We will see how, given two fixed points  $F, F'$  and a segment  $XY$  of length greater than  $FF'$  we can construct a point  $Q$  such that  $FQ + QF' = FF'$  and hence construct the ellipse as the locus of  $Q$  – provided we can find some suitable independent variable  $P$  and domain for it. If we make the domain of  $P$  the circle created using the Compass tool with  $F$  as centre and  $XY$  as radius, we need to find  $Q$  on  $FP$  so that  $FQ = QP$  i.e. form an isosceles triangle with  $F'P$  as base. So  $Q$  is the intersection of the perpendicular bisector of  $F'P$  with  $FP$ , and its locus is the ellipse. This method is reputed to be the ‘gardener’s method’ for making an elliptic flower bed using two sticks and a length of rope. **fig\_50**

The second locus method for the ellipse is actually the basis of a drawing instrument, called the ‘elliptic trammel’ which was designed for use by draughtsmen. Here we have two points  $P$  and  $Q$  which can slide on the major and minor axis respectively so that  $PQ$  is always a fixed distance  $OA$ . The locus of any point  $R$  on the line  $PQ$  should also be an ellipse. **fig\_51**

Can you see how to construct this for yourself? (Hint: use the Compass tool. Note that a 2nd point  $R'$  has had to be constructed to get the lower half of the ellipse.) Can you picture what will happen to the ellipse if you slide  $R$  on  $PQ$ ? What should it look like when  $R$  is the mid-point of  $PQ$ ?

Oddly enough it turns out that you can still buy the equivalent of an elliptical trammel, but this time as a device to cut ellipses from sheets of material such as wood! <http://www.rockler.com/product.cfm?page=17282&filter=ellipse>

So that wraps up this section in as smooth a way as we can. Next we shall look at the computer aided design equivalent of a French curve. If you feel like doing some research of your own you might explore something about curves of constant curvature – like 20p and 50p coins. They can be used as wheels, and roll fine – the trouble is that their centres do not stay at the same height! Can you imagine borrowing the bicycle from the Chinese inventor in the photograph?



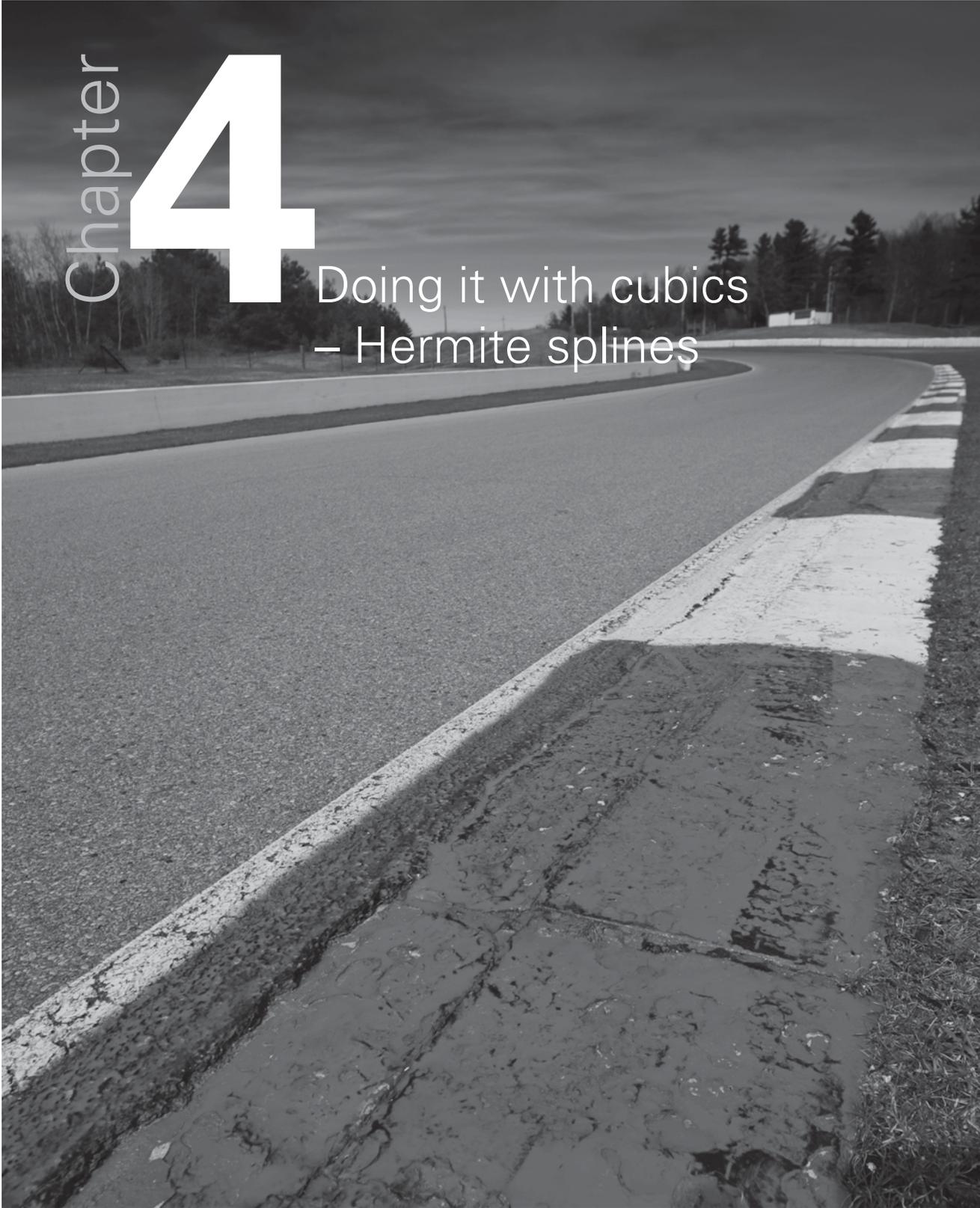
fig\_54



Chapter

# 4

Doing it with cubics  
– Hermite splines



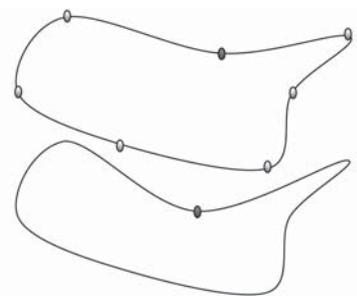
One of the techniques used in CAD is to produce a smooth shape controlled by just a few points. Suppose for example we want to design an interesting go-kart circuit. Assuming we want to make several laps, this will be a closed curve – and, for safety's sake, probably one that doesn't intersect itself! You can record a freehand curve using ICT in a number of ways e.g. by drawing with a stylus on a tablet or Interactive Whiteboard and using the built-in software to digitise the image – as a long list of pixel values. Similarly you could draw on paper and use a digital scanner to produce a similar result. You may have seen that such bitmap files can be very large. But they also are 'dead' in that once drawn we have no easy way of adjusting the shape, but would have to use a photo editor, or Windows Paint software, to erase and/or add pixels at a time – which is very tedious.

One feature of many drawing packages, including the last version of Microsoft Word in Office 2007, is the facility to create a smooth curve controlled by just a few mouse clicks. In Word, choose the Insert (2nd) tab, and then Shapes from the Illustrations tab. This produces a table of different tools. Look at those under the Lines heading and choose the 10th option called Curve whose symbol looks like a segment of a cubic curve with a maximum and minimum. Now you can click anywhere with the mouse to start drawing, and then each time you move and click again you get a segment of a smooth curve. Fifteen points were used to make the shape below, and when the 16th click was made on, or very close to, the 1st point, the software closes up the curves and gives you a finished graphic object. Try it for yourself. If you right-click in the graphic area, the 4th menu offering is Edit Points, and you will see exactly where the 15 points are located. You can move points, delete them, add more ones etc.. So that tool has the features we would like to have in designing the go-kart track.



fig\_55

The design **fig\_56** uses just 7 different points, and closes back to the starting point. So the curve is actually made up from 7 curved arcs each of which joins its neighbours smoothly at the end points. By changing the position of just one control point (the red blob), shown by the before and after closed curves, we have **local control**. This means that only a few of the arcs nearest to that control point are affected.



fig\_56

Each arc is uniquely determined by the position of its end-points and the tangent vectors at the joins with its neighbouring arcs. Instead of using the graphs of seven cubic functions it gives more flexibility if we represent each arc by a **parametric curve** – with both  $x$  and  $y$  as polynomials in the parameter  $t$ .

## The mathematical detail

(apparently not for the faint hearted, but not really that hard!)

Each polynomial will have just 4 degrees of freedom – since it must match the  $x$ -coordinates of its two end-points, and also match the gradients at its end points with the  $x$ -components of the tangent there. So we can look for a cubic polynomial like  $x(t) = p + qt + rt^2 + st^3$  – and arrange for  $t=0$  to correspond with the starting point A, and  $t=1$  with the finishing point B – so the domain of  $t$  for the arc AB is the interval  $0 \leq t \leq 1$ . The nice thing about cubic functions is that they can include inflections (i.e. wiggles)!

Since the arc joins points A ( $ax, ay$ ) and B ( $bx, by$ ) and has tangents with components  $[atx, aty]$  and  $[btx, bty]$  at A and B, then we should be able to make life easier by writing the equation of the arc in the special form:

$$x(t) = p(t).ax + q(t).bx + r(t).atx + s(t).btx, \quad y(t) = p(t).ay + q(t).by + r(t).aty + s(t).bty, \quad 0 \leq t \leq 1$$

In this case we will look for four cubic polynomials for our coefficients, rather than four constants. Then both  $x(t)$  and  $y(t)$  will be **linear combinations** of the four basic polynomials and every arc will have a similar form – all we have to do is to change the values like  $ax$ ,  $bx$ ,  $atx$  and  $btx$ . In fact these four basic cubic functions of  $t$  are called the **Hermite cubic spline** curves after the French mathematician Charles Hermite (1822-1901). You can find out more about them at: [http://en.wikipedia.org/wiki/Cubic\\_Hermite\\_spline](http://en.wikipedia.org/wiki/Cubic_Hermite_spline). We will now find equations for each of them.

If we take  $t = 0$  then we should be at A, and then  $x(0) = ax$  and  $y(0) = ay$ .

A simple way to ensure this is if  $p(0) = 1$  and  $q(0) = r(0) = s(0) = 0$ .

Can you see why  $t$  must then be a factor of  $q(t)$ ,  $r(t)$  and  $s(t)$ ? Cunning, eh?

Also when  $t = 1$  we are at B, so  $x(1) = bx$ .

Again we would like  $q(1) = 1$  and  $p(1) = r(1) = s(1) = 0$ .

In this case  $(1-t)$  must be a factor of  $p(t)$ ,  $r(t)$  and  $s(t)$  – agreed?

Differentiating with respect to  $t$  and writing e.g.  $\frac{dx(t)}{dt} = x'(t)$  we have:

$$x'(t) = p'(t).ax + q'(t).bx + r'(t).atx + s'(t).btx \quad \text{and similarly for } y'(t).$$

Again we look what happens when  $t = 0$ , and when  $t = 1$ .

Since  $x'(0) = atx$  we need  $r'(0) = 1$  and  $p'(0) = q'(0) = s'(0) = 0$ .

So  $t$  must be a factor of  $p'(t)$ ,  $q'(t)$ ,  $s'(t)$ .

Also  $x'(1) = btx$ , so  $s'(1) = 1$  and  $p'(1) = q'(1) = r'(1) = 0$  and  $(1-t)$  must be a factor of  $p'(t)$ ,  $q'(t)$ ,  $r'(t)$ .



So we find that  $d = 2$ ,  $e = 2$ ,  $f = 1$  and  $g = -1$  agreeing with the equations given in Wikipedia!

Plotting graphs of each of the functions:

$$p(t) = (1+2t)(1-t)^2$$

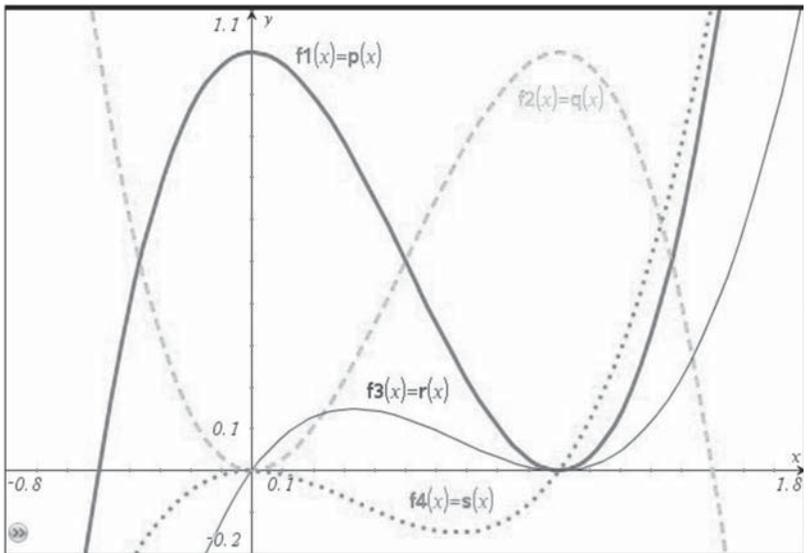
$$q(t) = (3-2t)t^2$$

$$r(t) = t(1-t)^2$$

$$s(t) = -t^2(1-t)$$

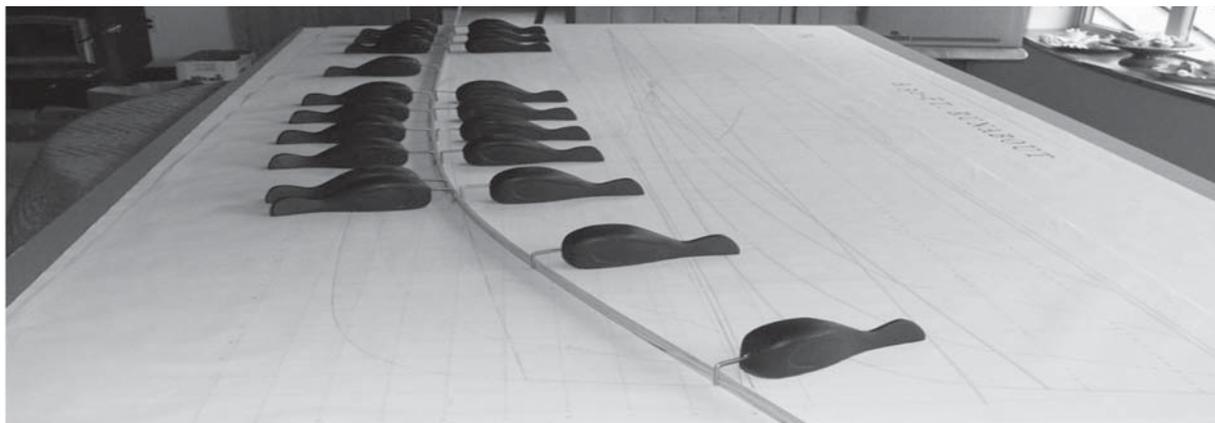
we can also see the behaviour in the interval  $0 \leq t \leq 1$  again in agreement with Wikipedia.

So armed with our 4 basic cubic splines we are ready to go forth and draw mathematical flexi-curves. **fig\_58**



**fig\_58**

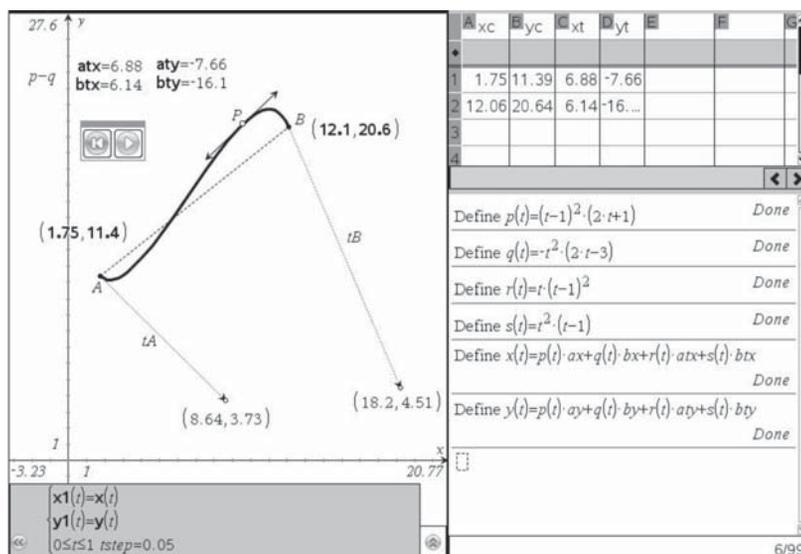
But what is a **spline**? The term dates from the days when draughtsmen made detailed technical drawings of objects on a large drawing board. To “fair a smooth curve” they had long strips of steel or laminated wood, and sets of heavy weights, called ‘ducks’. By adjusting the number and position of the ducks they could make the spline follow the desired path, which could then be traced. By recording the position of each duck they could ensure that they could reproduce the curve if needed. So the challenge to the mathematician was to produce the mathematical equivalent of the physical spline. **fig\_59**



fig\_59

The file Hermite.tns shows how we can start to build up each segment of a curve using 8 pieces of data: the coordinates of the ends A,B and the components of the tangent vectors  $t_A$  and  $t_B$ . The points and vectors are constructed in a Graphs page.

Using Coordinates and Equations we can find the coordinates of the 4 points used. With a right-click on the x-coordinate of A we can store it as the variable  $ax$ , and similarly we can define  $ay$ ,  $bx$  and  $by$ . Writing the text "p-q" on the screen we can use Calculate to feed it in turn with the values of the coordinates of the beginning and end of each vector to calculate the tangent components, which we store as  $atx$ ,  $aty$ ,  $btx$  and  $bty$ . These values are copied into the Spreadsheet, where e.g. A1 =  $ax$ , B1 =  $ay$ , C1 =  $atx$  and D1 =  $aty$ . The basic cubic splines are defined in the Calculator window, and used to define the parametric equations for the arc AB. This leaves A in the direction of the tangent  $t_A$ , and enters B in the direction of the tangent  $t_B$ . For the configuration shown, the arc has an inflection. Using 'Point On', the point P is constructed on the arc and its Tangent is also constructed. Using a right-click on P, its Attributes can be adjusted to give an animation speed of 3 unidirectionally along AB. Click on the green Play button to start the animation. **fig\_60**



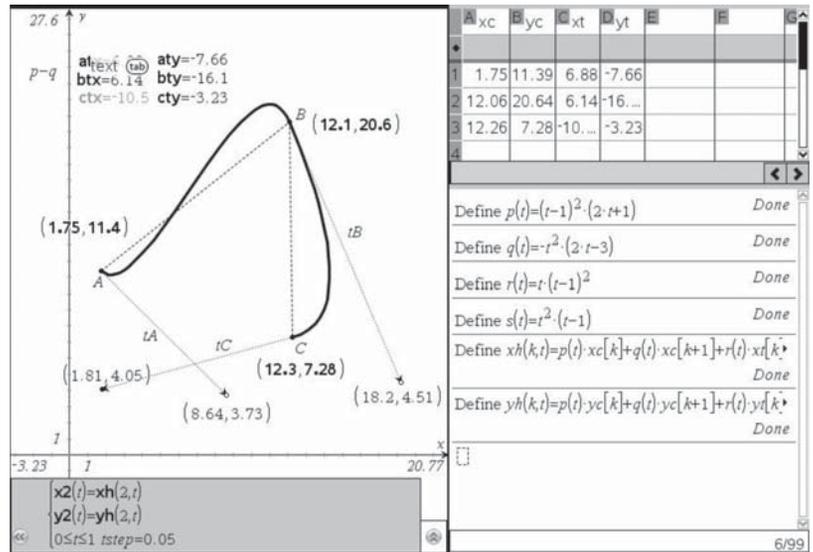
fig\_60

For a single arc, the spreadsheet is not actually needed. But if each new point has its own row, then the data for the 4th arc, say, will come from rows 4 and 5. So instead of referring to  $dx$  and  $ex$ , say, we can call them  $xc[4]$  and  $xc[5]$  – and that allows us to write the general form of the parametric equations using lists as, say:

$$xh(k,t) = p(t).xc[k] + q(t).xc[k+1] + r(t).xt[k] + s(t).xt[k+1]$$

$$yh(k,t) = p(t).yc[k] + q(t).yc[k+1] + r(t).yt[k] + s(t).yt[k+1].$$

So now we can see the effect of adding an extra point C and tangent  $tC$ . **fig\_61**



fig\_61

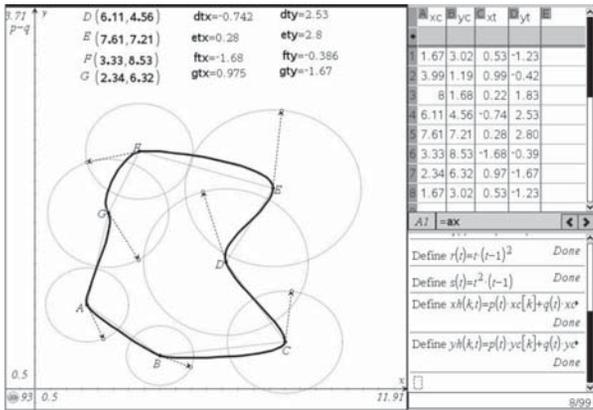
The important things about this model for a flexi-curve are that:

- (a) the finished curve passes through all the control points (so it interpolates the points);
- (b) the curved arcs join smoothly at the control points, having a common tangent;
- (c) the arcs themselves are also smooth, and flexible enough to allow for inflections.

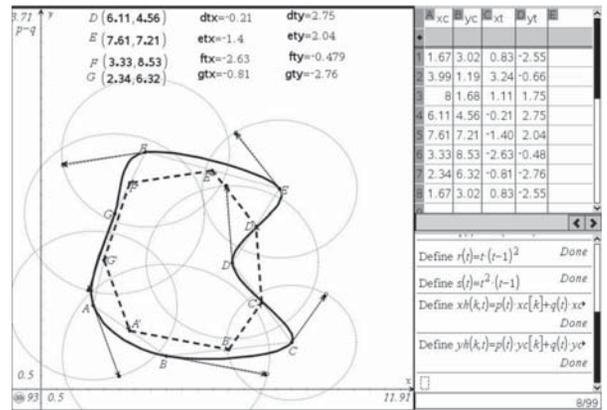
So here is a more complex version using the same principle but having seven control points. In order to create a closed curve, an 8th row has been added which is a duplicate of the first row.

The user interface has been designed so that you can change the position of each control point, the direction of its vector and the magnitude of that vector independently.

So, even if the mathematical detail was a closed book, you now have a modelling tool to work with! **fig\_62**

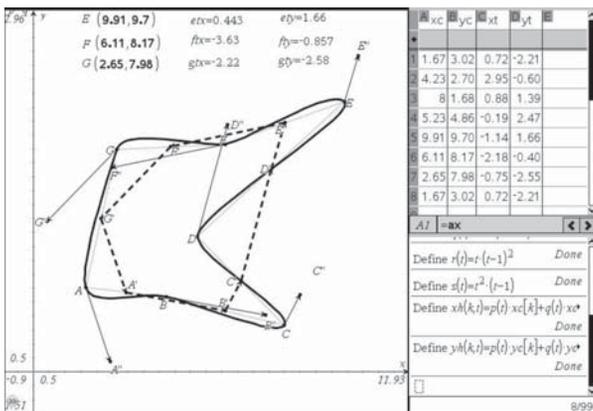


fig\_62

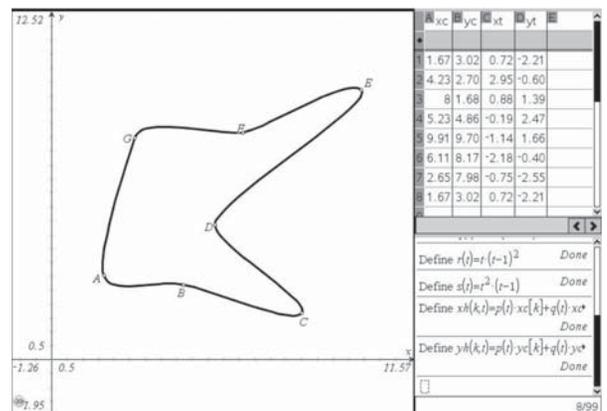


fig\_63

However, the flexi-curve used in Word only required the user to specify the control points, not the vectors. So it must use some technique (or algorithm) to calculate the vectors based on the geometry of the control points. We can only guess what this is, but the underlying polygon ABCDEFG might yield a clue. Suppose we find the midpoints of each segment e.g. let A' be the midpoint of AB etc.. Then for the points ABC, the vector A'B' might be a reasonable choice for the direction vector at B, and so. Let's test this out. We can construct each midpoint vector like A'B' and use it to translate each control point like B to B''. Then the vector BB'' will be a reasonable choice for the tangent at B. So we can drag our test points to lie over the new positions, like B'', to see if we get a reasonable shape. **fig\_63**



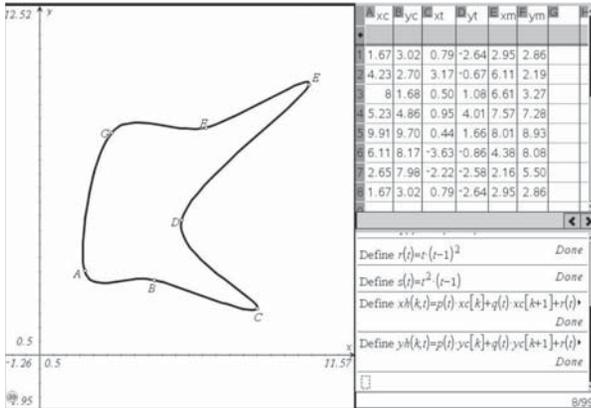
fig\_64



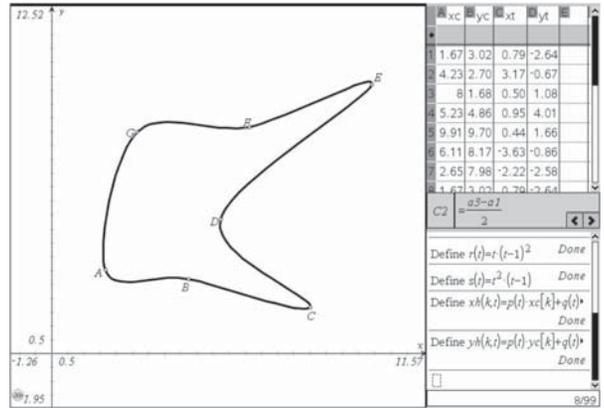
fig\_65

Now that looks to be OK we can redefine our old 'compass' vectors to have their endpoints at the new B'' style positions. **fig\_64**

Now that's worked OK, we can use Hide/Show to tidy things up. **fig\_65**



fig\_66



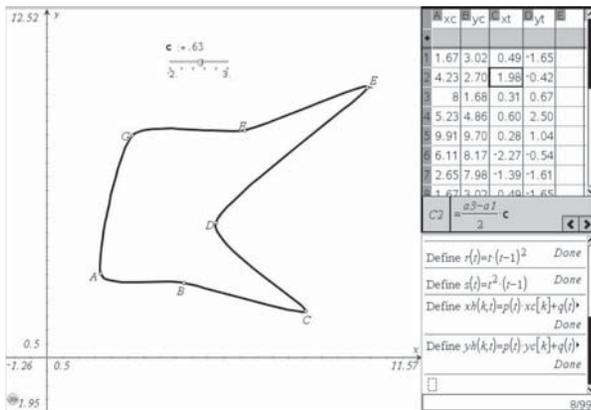
fig\_67

The problem with redefining the points to define the vectors is that now we have lost the means of changing the stored variables *atx*, *aty* etc.. So changing positions of the control points is no longer updating the spreadsheet as we would like. But that's no problem! Given the coordinates of the control points stored in the columns *xc* and *yc* of the spreadsheet we should be able to compute all we need. As a first step we can introduce additional columns *xm* and *ym* to hold the coordinates of the midpoints like *A*. Then using these we can calculate the vector components *xt* and *yt*. **fig\_66**

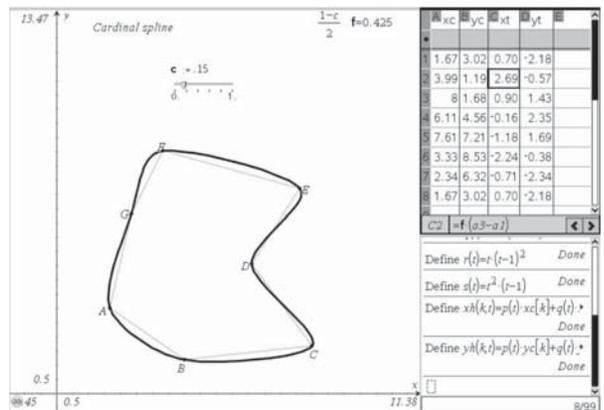
However, we can also find the values for the cells in *xt* and *yt* directly from those in *xc* and *yc* without the explicit calculations for *xm* and *ym*: **fig\_67**

Consider cell C2 =  $e2 - e1 = (a2+a3)/2 - (a1+a2)/2 = (a3-a1)/2$

So that's the typical formula, but we need to handle the 'wraparounds':  
 $C1 = (a2-a7)/2 = C8$ .



fig\_68



fig\_69

Now we effectively have the equivalent of the Word flexi-curve in the Graphs page, but also in the Lists & Spreadsheets and the Calculator views we also have on show the algorithm that Word is using behind the scenes! As you move any control-point you should now see just a few of the spreadsheet entries changing – showing 'local control'. **fig\_68**

The Wikipedia mentions something called the 'cardinal spline'. In fact we have already just done this for the case  $c = 1$  – and we can easily model the general case by adjusting values within the spreadsheet. We don't have to do what we're told! So the slider has been set up to explore the effects of values of  $c$  between -2 and 3.

Now we can also explore changing the tension parameter  $c$ . **fig\_69**

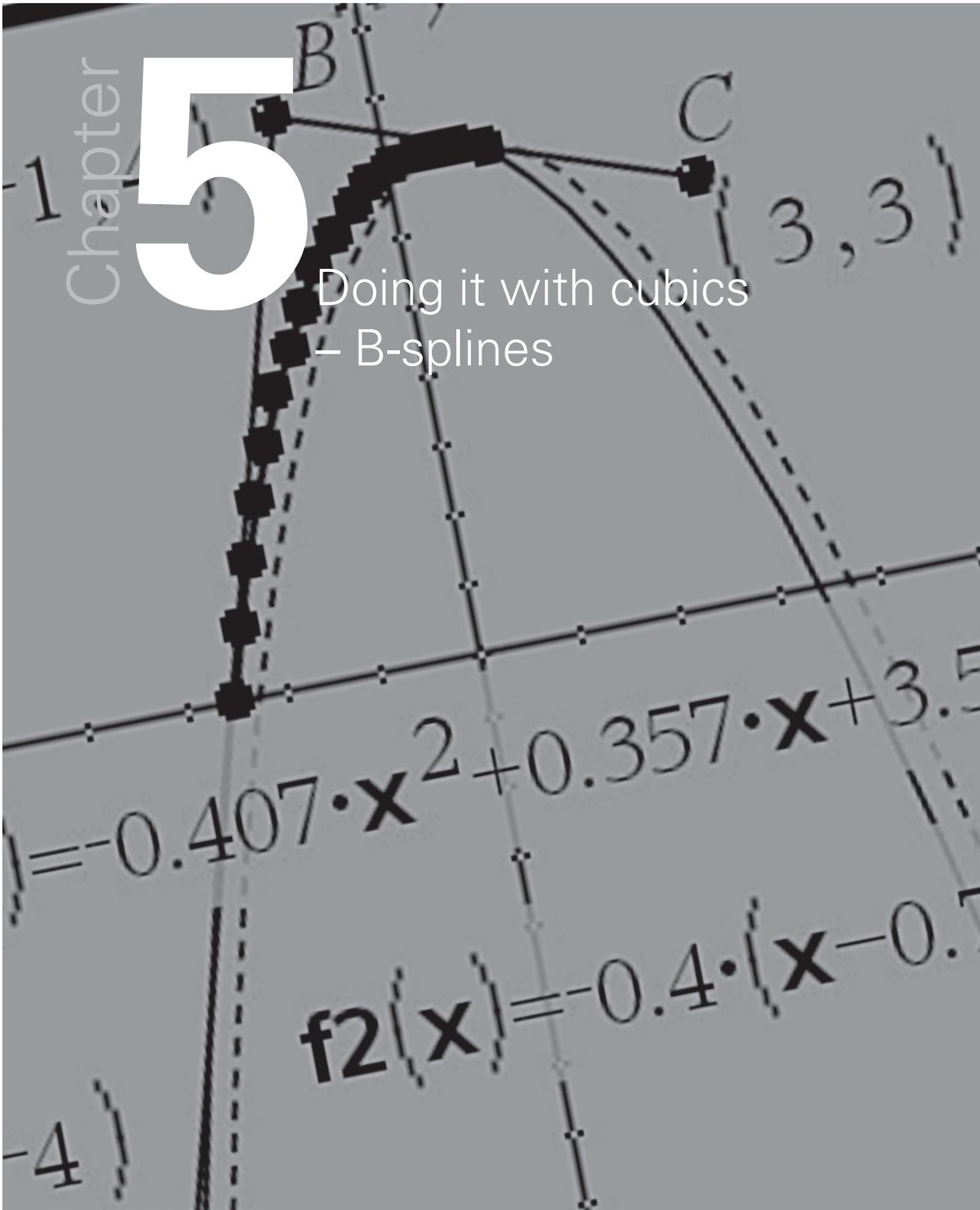
### Here are some things you might like to follow up

- Ex. 1:** Make an interesting go-kart circuit using 7 points.
- Ex. 2:** Trace round a shoe (or use an insole) and see if you can model the resulting shape using just 7 points.
- Ex. 3:** Create one or more additional points and vectors, also creating extra variables, rows of the spreadsheet and parametric arcs and use it to design more complex curves.
- Ex. 4:** Find out more from the Wikipedia reference (and by your own searches) about other ways of computing the vectors from the coordinates to give the equivalent of the way Word draws the curves without being told the tangent vectors and write or adapt TI-Nspire files to investigate them.
- Ex.5:** Find out more about Edwin Catmull and his mathematical contributions to the world of film and video.
- Ex.6:** Find out about the radius of curvature of a curve, and explore whether or not our cubic Hermite splines always give the same radius of curvature at each side of a control point. This would be an important consideration in designing roads, railways etc.. Can you find out (or maybe invent) a technique for the calculation of the vectors at the joins to give continuity of position, tangent **and** curvature?
- Ex.7:** Find out how to compute the length of a parametric arc between  $t = 0$  and  $t = 1$  and hence compute the total arc length of a closed cubic-spline curve.

Chapter

## 5

Doing it with cubics  
– B-splines



We've already met most of the key ideas we need - B-splines just provide a neat way of sewing them together. As with the Hermite curves, we will make a smooth curve by joining together a set of cubic arcs. We will also use four pieces of information, but this time we will use the coordinates of four successive points like A, B, C and D. The x-coordinates of a point T on the arc will be a weighted average (a blend) of the four x-coordinates, and similarly for the y-coordinates. As with Hermite, the curve could be in 3D (or higher dimensions) by taking z-coordinates as well. Also the curve could be open or joined up (as with a race-track). We just need to know what the weighting functions will be. While they are not quite as obvious as the Bézier weighting functions they are relatively straightforward polynomial functions.

First, take a step back and see what a quadratic arc formed by 3 points A, B and C looks like. The functions we are going to use are:

$$f(t) = \frac{1}{2} (1 - t)^2$$

$$g(t) = \frac{1}{2} + t(1-t)$$

$$h(t) = \frac{1}{2} t^2$$

where the parameter, t, runs from 0 to 1. Check that  $f(t) + g(t) + h(t) = 1$  for all values of t.

We will also take the opportunity to see first how we can make an algorithm explicit in a spreadsheet.

If A has coordinates (ax, ay) etc., then the coordinates of any point T will be given by the equations:

$$tx = f(t) ax + g(t) bx + h(t) cx$$

$$ty = f(t) ay + g(t) by + h(t) cy$$

We just need to convert the algebraic formula into a version the spreadsheet "understands".

If d3 represents the contents of the cell D3, we can use the "\$" symbol to fix either the column reference "d" or the row reference "3" or both: e.g. \$d3, d\$3 or \$d\$3. These notations are called absolute, rather than relative, references and are important when filling cells across a row or down column.

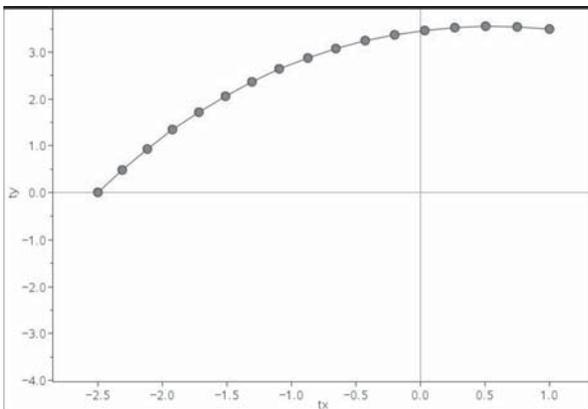
Here is a spreadsheet set up to generate 17 points on the quadratic arc with values of  $t$  from 0 to 1 in increments of  $1/16$  (as with a For-Next loop in a programming language). **fig70**

The formula in cell G1 is:  $g1 = \$d1*a\$1 + \$e1*a\$2 + \$f1*a\$3$   
 When copied down this becomes:  $g2 = \$d2*a\$1 + \$e2*a\$2 + \$f2*a\$3$   
 And when copied across we get:  $h1 = \$d1*b\$1 + \$e1*b\$2 + \$f1*b\$3$

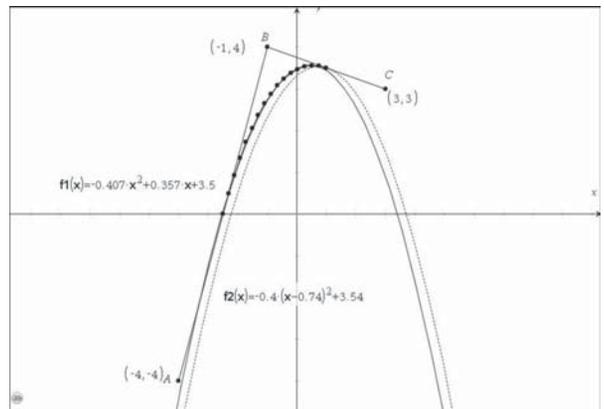
A	B	C	D	E	F	G	H	I	J	K	L
x	y	t	f	g	h	tx	ty				
			$=1/2*(1-t)^2$	$=1/2+t*(1-t)$	$=1/2*t^2$						=QuadRe
1	-4	-4	0	1/2	1/2	0	-5/2	0			Title Quadra...
2	-1	4	0.0625	0.439453	0.558594	0.0019...	-2.31055	0.4824...			RegEqn...a*x^2+b...
3	3	3	0.125	0.382813	0.609375	0.0078...	-2.11719	0.9296...			a -0.4066...
4			0.1875	0.330078	0.652344	0.0175...	-1.91992	1.3418			b 0.3566...
5			0.25	0.28125	0.6875	0.03125	-1.71875	1.71875			c 3.495
6			0.3125	0.236328	0.714844	0.0488...	-1.51367	2.06055			R^2 0.9992...
7			0.375	0.195313	0.734375	0.0703...	-1.30469	2.36719			Resid {-0.062...
8			0.4375	0.158203	0.746094	0.0957...	-1.0918	2.63867			
9			0.5	0.125	0.75	0.125	-0.875	2.875			
10			0.5625	0.095703	0.746094	0.1582...	-0.6542...	3.07617			
11			0.625	0.070313	0.734375	0.1953...	-0.4296...	3.24219			
12			0.6875	0.048828	0.714844	0.2363...	-0.2011...	3.37305			
13			0.75	0.03125	0.6875	0.28125	0.03125	3.46875			
14			0.8125	0.017578	0.652344	0.3300...	0.2675...	3.5293			
15			0.875	0.007813	0.609375	0.3828...	0.5078...	3.55469			
16			0.9375	0.001953	0.558594	0.4394...	0.7519...	3.54492			
17			1.	0.	0.5	0.5	1.	3.5			
18											

fig\_70

We can display the results graphically in either a Data & Statistics page or a Graphs page.



fig\_71



fig\_72

The arc looks quadratic, but in the sense of a rotated parabola rather than either a quadratic regression function or a manipulated quadratic function. Try dragging A, B and/or C around so that the axis of the arc is (nearly) horizontal, or is diagonally inclined. Hence we cannot, in general, express the curve as a simple function of  $x$ , but our equations for  $T$  ( $tx, ty$ ) define the locus of  $T$  in terms of the graph of parametric equations.

We can perform the whole task in a Graphs page. First three points A, B and C have been constructed, their coordinates displayed, and each coordinate stored in a variable, such as ax. Now the three functions  $f_1(x)$ ,  $f_2(x)$  and  $f_3(x)$  can be defined as Function Graph Plots, but then their graphs are hidden. Finally we can define a Parametric Graph Plot by the function displayed on the screen. In the definition screen we also fixed the value of the parameter  $t$  to be between 0 and 1 with an increment of  $1/16$ . Check that you can now move A, B and C around and explain the geometric relationship between the quadratic parametric arc and the triangle ABC.

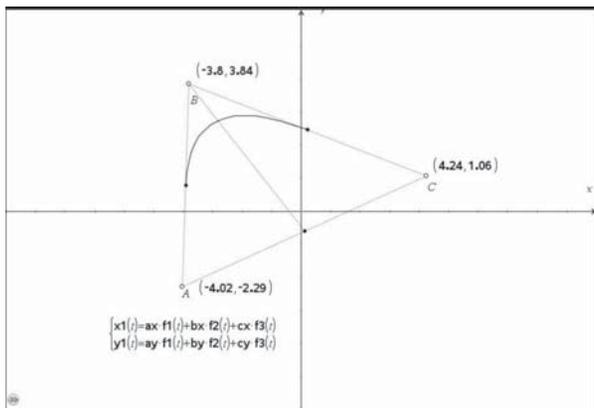
Having gone back to quadratics we can now step forward again to cubics. We just need an extra point D and an extra weighting function  $f_4(x)$ . We also need new definitions for all four weighting functions which are:

$$f_1(t) = 1/6 (1 - t)^3$$

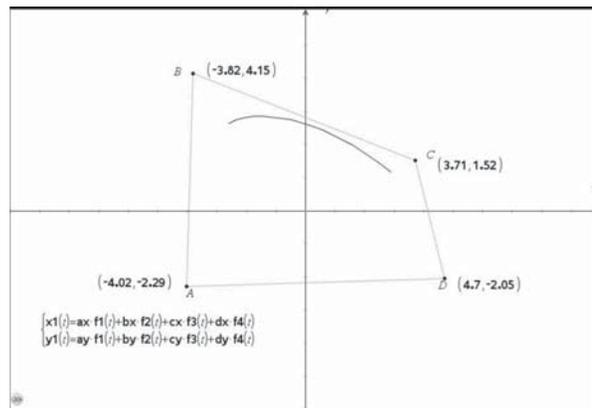
$$f_2(t) = 1/6 ((2 - t)^3 - 4(1 - t)^3)$$

$$f_3(t) = 1/6 ((1 + t)^3 - 4t^3)$$

$$f_4(t) = 1/6 t^3$$



fig\_73

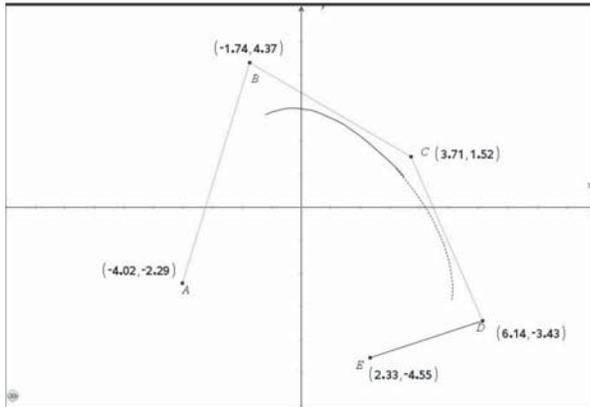


fig\_74

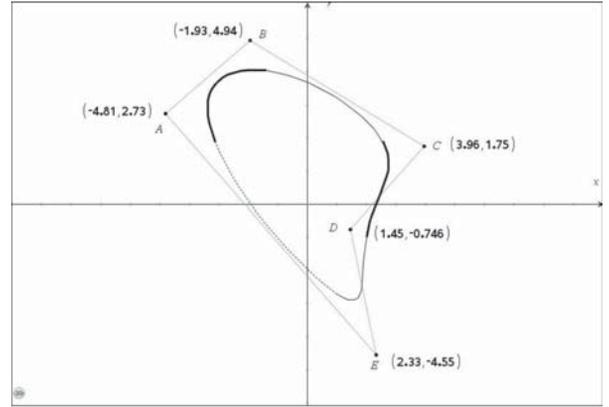
Now we see how the cubic arc is related to the quadrilateral defined by the four control points A, B, C and D.

So, if we have, say, 5 points we can find a first arc controlled by ABCD, then a second arc controlled by BCDE etc.. Or we could complete a closed arc in the quadrilateral by computing arcs controlled by ABCD, BCDA, CDAB and DABC etc..

So now you have been inducted into the dark secrets of the computational geometry which has been developed over many years initially to aid designers and engineers, but now increasingly in the entertainment industry for cartoons, animations and computer games.



fig\_75



fig\_76

We have not (yet) gone into techniques for using colour, such as rendering, nor to explore the wide world of three dimensions. They will have to wait for a future book – watch this space!



## STEM activities with TI-Nspire

The coming decade will see an increasing need for a flexible work force possessing a wide range of skills in Science, Technology, Engineering and Mathematics (STEM) in order to meet the needs of the new high level industries and to be able to produce technologically complex products. The booklets in this series were developed with this need in mind. They aim to provide stimulating activities which link aspects of STEM and at the same time encourage the use of technology and an awareness of its potential.

## Why use TI-Nspire for STEM subjects?

TI-Nspire provides a learning platform which dynamically links a variety of ICT applications including documents, graphs, geometry, statistics, spreadsheets, data logging and a calculator. This dynamic linking assists students in making connections not only between different areas of mathematics but also with other areas of the curriculum and STEM subjects in particular.

There is now a wealth of data available on the Internet in a variety of forms which can be copied directly into TI-Nspire. Data can also be captured by linking to a variety of data logging devices such as motion sensors or by recording manually or using video analysis. It is then possible to manipulate, display and analyse this data in a variety of ways or to try to model the data using a variety of data handling and function plotting tools.

## About the booklets

All the activities contain some background scientific or other information together with links to appropriate websites. Many of the activities are suitable for a range of ages and aptitudes with more challenging ideas being suggested as extension activities. Further information and TI-Nspire files for the activities can be found at [www.nspiringlearning.org.uk](http://www.nspiringlearning.org.uk).

### There are five booklets in the series:

**Introduction** - contains a brief introduction and instructions for getting started using some of the features of TI-Nspire that are used frequently in the other booklets.

**Capturing data: Modelling and Interpretation** - contains activities which use a variety of data logging probes to collect real data and analyse it further.

**Using real world data** - contains activities which investigate and analyse data in a variety of ways from ready-made or easily generated data sets in a variety of contexts.

**Mathematics in motion** - contains investigations into modelling motion based on different forms of data collection: manual, video and data logging.

**Computer graphics** - brings together a number of different mathematical approaches which are used by computer programmers to create and manipulate images; techniques that have now found their way into "virtual reality". These activities make use of TI-Nspire applications especially Graphs and Geometry.

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